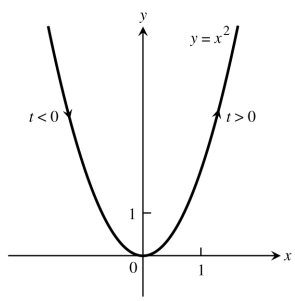
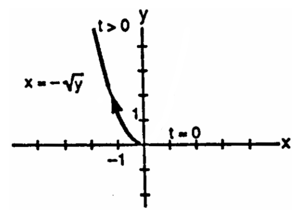
CHAPTER 11 PARAMETRIC EQUATIONS AND POLAR COORDINATES

11.1 PARAMETRIZATIONS OF PLANE CURVES

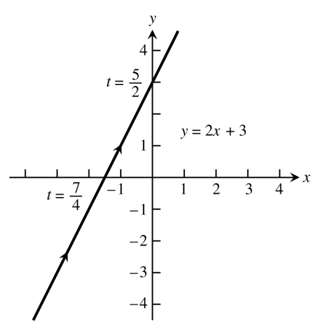
1. $x = 3t$, $y = 9t^2$, $-\infty < t < \infty \Rightarrow y = x^2$



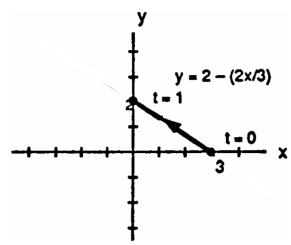
2. $x = -\sqrt{t}$, $y = t$, $t \geq 0 \Rightarrow x = -\sqrt{y}$ or $y = x^2$, $x \leq 0$



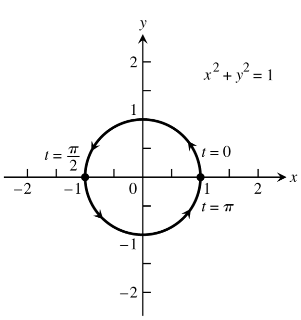
3. $x = 2t - 5$, $y = 4t - 7$, $-\infty < t < \infty$
$\Rightarrow x + 5 = 2t \Rightarrow 2(x + 5) = 4t$
$\Rightarrow y = 2(x + 5) - 7 \Rightarrow y = 2x + 3$



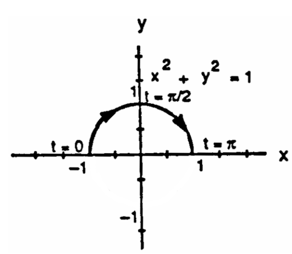
4. $x = 3 - 3t$, $y = 2t$, $0 \leq t \leq 1 \Rightarrow \frac{y}{2} = t$
$\Rightarrow x = 3 - 3\left(\frac{y}{2}\right) \Rightarrow 2x = 6 - 3y$
$\Rightarrow y = 2 - \frac{2}{3}x$, $0 \leq x \leq 3$



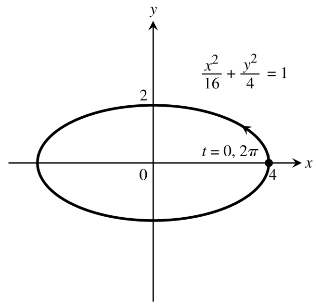
5. $x = \cos 2t$, $y = \sin 2t$, $0 \leq t \leq \pi$
$\Rightarrow \cos^2 2t + \sin^2 2t = 1 \Rightarrow x^2 + y^2 = 1$



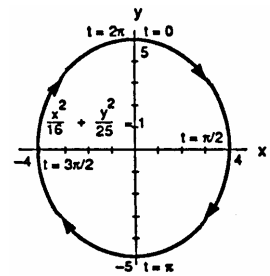
6. $x = \cos(\pi - t)$, $y = \sin(\pi - t)$, $0 \leq t \leq \pi$
$\Rightarrow \cos^2(\pi - t) + \sin^2(\pi - t) = 1$
$\Rightarrow x^2 + y^2 = 1$, $y \geq 0$



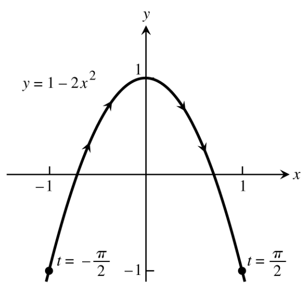
7. $x = 4\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$
$\Rightarrow\ \frac{16\cos^2 t}{16} + \frac{4\sin^2 t}{4} = 1\ \Rightarrow\ \frac{x^2}{16} + \frac{y^2}{4} = 1$



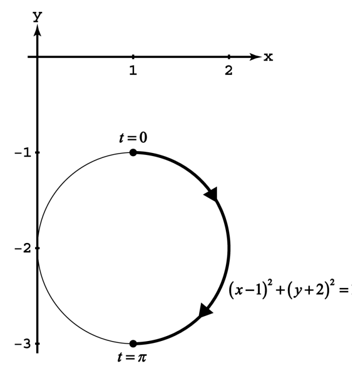
8. $x = 4\sin t,\ y = 5\cos t,\ 0 \le t \le 2\pi$
$\Rightarrow\ \frac{16\sin^2 t}{16} + \frac{25\cos^2 t}{25} = 1\ \Rightarrow\ \frac{x^2}{16} + \frac{y^2}{25} = 1$



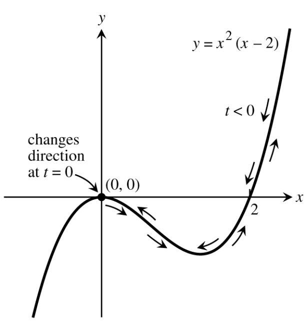
9. $x = \sin t,\ y = \cos 2t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$
$\Rightarrow\ y = \cos 2t = 1 - 2\sin^2 t \Rightarrow y = 1 - 2x^2$



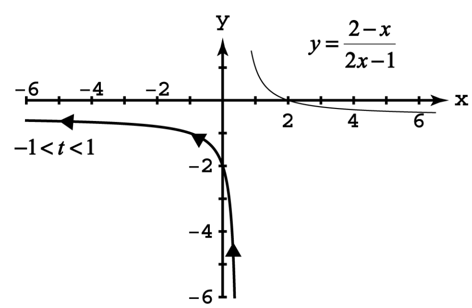
10. $x = 1 + \sin t,\ y = \cos t - 2,\ 0 \le t \le \pi$
$\Rightarrow\ \sin^2 t + \cos^2 t = 1\ \Rightarrow\ (x-1)^2 + (y+2)^2 = 1$



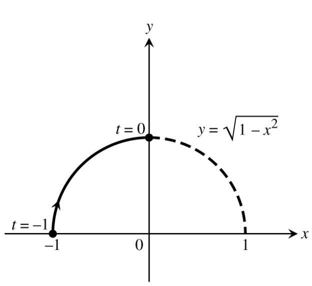
11. $x = t^2,\ y = t^6 - 2t^4,\ -\infty < t < \infty$
$\Rightarrow y = (t^2)^3 - 2(t^2)^2 \Rightarrow y = x^3 - 2x^2$



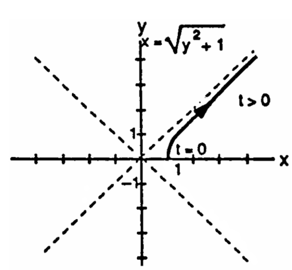
12. $x = \frac{t}{t-1},\ y = \frac{t-2}{t+1},\ -1 < t < 1$
$\Rightarrow\ t = \frac{x}{x-1}\ \Rightarrow y = \frac{2-x}{2x-1}$



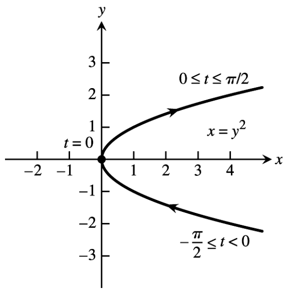
13. $x = t,\ y = \sqrt{1-t^2},\ -1 \le t \le 0$
$\Rightarrow\ y = \sqrt{1-x^2}$



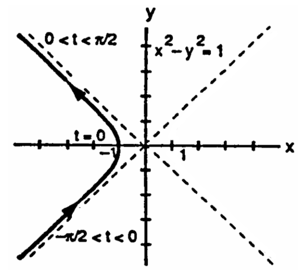
14. $x = \sqrt{t+1},\ y = \sqrt{t},\ t \ge 0$
$\Rightarrow\ y^2 = t\ \Rightarrow\ x = \sqrt{y^2+1},\ y \ge 0$



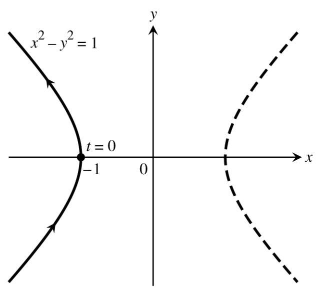
15. $x = \sec^2 t - 1$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$
$\Rightarrow \sec^2 t - 1 = \tan^2 t \Rightarrow x = y^2$



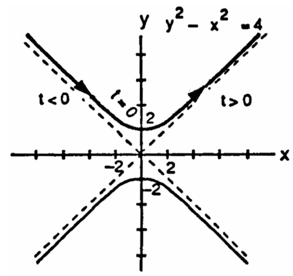
16. $x = -\sec t$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$
$\Rightarrow \sec^2 t - \tan^2 t = 1 \Rightarrow x^2 - y^2 = 1$



17. $x = -\cosh t$, $y = \sinh t$, $-\infty < 1 < \infty$
$\Rightarrow \cosh^2 t - \sinh^2 t = 1 \Rightarrow x^2 - y^2 = 1$



18. $x = 2\sinh t$, $y = 2\cosh t$, $-\infty < t < \infty$
$\Rightarrow 4\cosh^2 t - 4\sinh^2 t = 4 \Rightarrow y^2 - x^2 = 4$



19. (a) $x = a\cos t$, $y = -a\sin t$, $0 \le t \le 2\pi$
(b) $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$
(c) $x = a\cos t$, $y = -a\sin t$, $0 \le t \le 4\pi$
(d) $x = a\cos t$, $y = a\sin t$, $0 \le t \le 4\pi$

20. (a) $x = a\sin t$, $y = b\cos t$, $\frac{\pi}{2} \le t \le \frac{5\pi}{2}$
(b) $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$
(c) $x = a\sin t$, $y = b\cos t$, $\frac{\pi}{2} \le t \le \frac{9\pi}{2}$
(d) $x = a\cos t$, $y = b\sin t$, $0 \le t \le 4\pi$

21. Using $(-1, -3)$ we create the parametric equations $x = -1 + at$ and $y = -3 + bt$, representing a line which goes through $(-1, -3)$ at $t = 0$. We determine a and b so that the line goes through $(4, 1)$ when $t = 1$. Since $4 = -1 + a \Rightarrow a = 5$. Since $1 = -3 + b \Rightarrow b = 4$. Therefore, one possible parameterization is $x = -1 + 5t$, $y = -3 + 4t$, $0 \le t \le 1$.

22. Using $(-1, 3)$ we create the parametric equations $x = -1 + at$ and $y = 3 + bt$, representing a line which goes through $(-1, 3)$ at $t = 0$. We determine a and b so that the line goes through $(3, -2)$ when $t = 1$. Since $3 = -1 + a \Rightarrow a = 4$. Since $-2 = 3 + b \Rightarrow b = -5$. Therefore, one possible parameterization is $x = -1 + 4t$, $y = 3 - 5t$, $0 \le t \le 1$.

23. The lower half of the parabola is given by $x = y^2 + 1$ for $y \le 0$. Substituting t for y, we obtain one possible parameterization $x = t^2 + 1$, $y = t$, $t \le 0$.

24. The vertex of the parabola is at $(-1, -1)$, so the left half of the parabola is given by $y = x^2 + 2x$ for $x \le -1$. Substituting t for x, we obtain one possible parametrization: $x = t$, $y = t^2 + 2t$, $t \le -1$.

25. For simplicity, we assume that x and y are linear functions of t and that the point(x, y) starts at $(2, 3)$ for $t = 0$ and passes through $(-1, -1)$ at $t = 1$. Then $x = f(t)$, where $f(0) = 2$ and $f(1) = -1$.
Since slope $= \frac{\Delta x}{\Delta t} = \frac{-1-2}{1-0} = -3$, $x = f(t) = -3t + 2 = 2 - 3t$. Also, $y = g(t)$, where $g(0) = 3$ and $g(1) = -1$.
Since slope $= \frac{\Delta y}{\Delta t} = \frac{-1-3}{1-0} = -4$. $y = g(t) = -4t + 3 = 3 - 4t$.
One possible parameterization is: $x = 2 - 3t$, $y = 3 - 4t$, $t \ge 0$.

26. For simplicity, we assume that x and y are linear functions of t and that the point(x, y) starts at $(-1, 2)$ for $t = 0$ and passes through $(0, 0)$ at $t = 1$. Then $x = f(t)$, where $f(0) = -1$ and $f(1) = 0$.
Since slope $= \frac{\Delta x}{\Delta t} = \frac{0-(-1)}{1-0} = 1$, $x = f(t) = 1t + (-1) = -1 + t$. Also, $y = g(t)$, where $g(0) = 2$ and $g(1) = 0$.
Since slope $= \frac{\Delta y}{\Delta t} = \frac{0-2}{1-0} = -2$. $y = g(t) = -2t + 2 = 2 - 2t$.
One possible parameterization is: $x = -1 + t$, $y = 2 - 2t$, $t \ge 0$.

27. Since we only want the top half of a circle, $y \ge 0$, so let $x = 2\cos t$, $y = 2|\sin t|$, $0 \le t \le 4\pi$

28. Since we want x to stay between -3 and 3, let $x = 3\sin t$, then $y = (3\sin t)^2 = 9\sin^2 t$, thus $x = 3\sin t$, $y = 9\sin^2 t$, $0 \le t < \infty$

29. $x^2 + y^2 = a^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$; let $t = \frac{dy}{dx} \Rightarrow -\frac{x}{y} = t \Rightarrow x = -yt$. Substitution yields $y^2t^2 + y^2 = a^2 \Rightarrow y = \frac{a}{\sqrt{1+t^2}}$ and $x = \frac{-at}{\sqrt{1+t}}$, $-\infty < t < \infty$

30. In terms of θ, parametric equations for the circle are $x = a\cos\theta$, $y = a\sin\theta$, $0 \le \theta < 2\pi$. Since $\theta = \frac{s}{a}$, the arc length parametrizations are: $x = a\cos\frac{s}{a}$, $y = a\sin\frac{s}{a}$, and $0 \le \frac{s}{a} < 2\pi \Rightarrow 0 \le s \le 2\pi a$ is the interval for s.

31. Drop a vertical line from the point (x, y) to the x-axis, then θ is an angle in a right triangle, and from trigonometry we know that $\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$. The equation of the line through $(0, 2)$ and $(4, 0)$ is given by $y = -\frac{1}{2}x + 2$. Thus $x\tan\theta = -\frac{1}{2}x + 2 \Rightarrow x = \frac{4}{2\tan\theta + 1}$ and $y = \frac{4\tan\theta}{2\tan\theta + 1}$ where $0 \le \theta < \frac{\pi}{2}$.

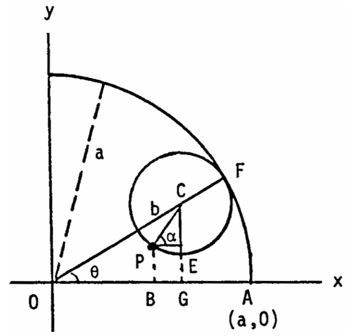
32. Drop a vertical line from the point (x, y) to the x-axis, then θ is an angle in a right triangle, and from trigonometry we know that $\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$. Since $y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow (x\tan\theta)^2 = x \Rightarrow x = \cot^2\theta \Rightarrow y = \cot\theta$ where $0 < \theta \le \frac{\pi}{2}$.

33. The equation of the circle is given by $(x-2)^2 + y^2 = 1$. Drop a vertical line from the point (x, y) on the circle to the x-axis, then θ is an angle in a right triangle. So that we can start at $(1, 0)$ and rotate in a clockwise direction, let $x = 2 - \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$.

34. Drop a vertical line from the point (x, y) to the x-axis, then θ is an angle in a right triangle, whose height is y and whose base is $x + 2$. By trigonometry we have $\tan\theta = \frac{y}{x+2} \Rightarrow y = (x+2)\tan\theta$. The equation of the circle is given by $x^2 + y^2 = 1 \Rightarrow x^2 + ((x+2)\tan\theta)^2 = 1 \Rightarrow x^2\sec^2\theta + 4x\tan^2\theta + 4\tan^2\theta - 1 = 0$. Solving for x we obtain
$$x = \frac{-4\tan^2\theta \pm \sqrt{(4\tan^2\theta)^2 - 4\sec^2\theta\,(4\tan^2\theta - 1)}}{2\sec^2\theta} = \frac{-4\tan^2\theta \pm 2\sqrt{1 - 3\tan^2\theta}}{2\sec^2\theta} = -2\sin^2\theta \pm \cos\theta\sqrt{\cos^2\theta - 3\sin^2\theta}$$
$= -2 + 2\cos^2\theta \pm \cos\theta\sqrt{4\cos^2\theta - 3}$ and $y = \left(-2 + 2\cos^2\theta \pm \cos\theta\sqrt{4\cos^2\theta - 3} + 2\right)\tan\theta$
$= 2\sin\theta\cos\theta \pm \sin\theta\sqrt{4\cos^2\theta - 3}$. Since we only need to go from $(1, 0)$ to $(0, 1)$, let
$x = -2 + 2\cos^2\theta + \cos\theta\sqrt{4\cos^2\theta - 3}$, $y = 2\sin\theta\cos\theta + \sin\theta\sqrt{4\cos^2\theta - 3}$, $0 \le \theta \le \tan^{-1}\left(\frac{1}{2}\right)$.
To obtain the upper limit for θ, note that $x = 0$ and $y = 1$, using $y = (x+2)\tan\theta \Rightarrow 1 = 2\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$.

35. Extend the vertical line through A to the x-axis and let C be the point of intersection. Then $OC = AQ = x$ and $\tan t = \frac{2}{OC} = \frac{2}{x} \Rightarrow x = \frac{2}{\tan t} = 2\cot t$; $\sin t = \frac{2}{OA} \Rightarrow OA = \frac{2}{\sin t}$; and $(AB)(OA) = (AQ)^2 \Rightarrow AB\left(\frac{2}{\sin t}\right) = x^2$
$\Rightarrow AB\left(\frac{2}{\sin t}\right) = \left(\frac{2}{\tan t}\right)^2 \Rightarrow AB = \frac{2\sin t}{\tan^2 t}$. Next $y = 2 - AB\sin t \Rightarrow y = 2 - \left(\frac{2\sin t}{\tan^2 t}\right)\sin t =$
$2 - \frac{2\sin^2 t}{\tan^2 t} = 2 - 2\cos^2 t = 2\sin^2 t$. Therefore let $x = 2\cot t$ and $y = 2\sin^2 t$, $0 < t < \pi$.

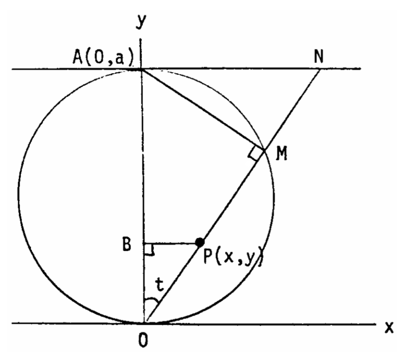
36. Arc PF = Arc AF since each is the distance rolled and $\frac{\text{Arc PF}}{b} = \angle FCP \Rightarrow \text{Arc PF} = b(\angle FCP)$; $\frac{\text{Arc AF}}{a} = \theta$ $\Rightarrow \text{Arc AF} = a\theta \Rightarrow a\theta = b(\angle FCP) \Rightarrow \angle FCP = \frac{a}{b}\theta$; $\angle OCG = \frac{\pi}{2} - \theta$; $\angle OCG = \angle OCP + \angle PCE$ $= \angle OCP + \left(\frac{\pi}{2} - \alpha\right)$. Now $\angle OCP = \pi - \angle FCP$ $= \pi - \frac{a}{b}\theta$. Thus $\angle OCG = \pi - \frac{a}{b}\theta + \frac{\pi}{2} - \alpha \Rightarrow \frac{\pi}{2} - \theta$ $= \pi - \frac{a}{b}\theta + \frac{\pi}{2} - \alpha \Rightarrow \alpha = \pi - \frac{a}{b}\theta + \theta = \pi - \left(\frac{a-b}{b}\theta\right)$.



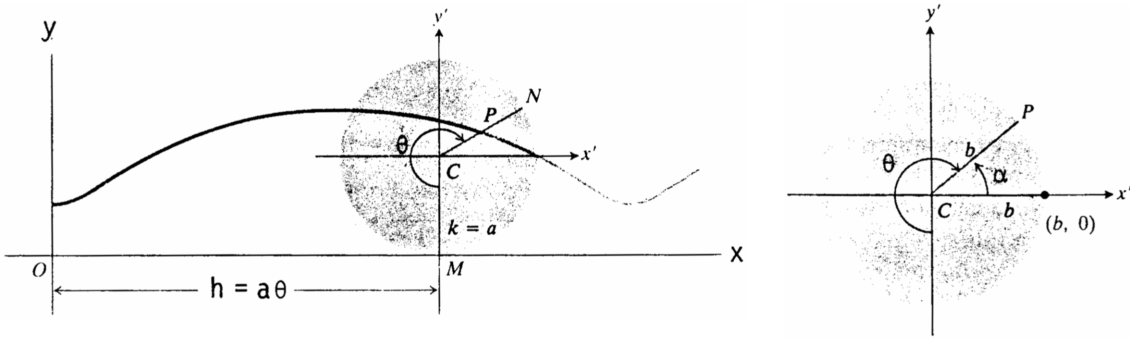
Then $x = OG - BG = OG - PE = (a-b)\cos\theta - b\cos\alpha = (a-b)\cos\theta - b\cos\left(\pi - \frac{a-b}{b}\theta\right)$ $= (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$. Also $y = EG = CG - CE = (a-b)\sin\theta - b\sin\alpha$ $= (a-b)\sin\theta - b\sin\left(\pi - \frac{a-b}{b}\theta\right) = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$. Therefore $x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$ and $y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$.

If $b = \frac{a}{4}$, then $x = \left(a - \frac{a}{4}\right)\cos\theta + \frac{a}{4}\cos\left(\frac{a - \left(\frac{a}{4}\right)}{\left(\frac{a}{4}\right)}\theta\right)$
$= \frac{3a}{4}\cos\theta + \frac{a}{4}\cos 3\theta = \frac{3a}{4}\cos\theta + \frac{a}{4}(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta)$
$= \frac{3a}{4}\cos\theta + \frac{a}{4}((\cos\theta)(\cos^2\theta - \sin^2\theta) - (\sin\theta)(2\sin\theta\cos\theta))$
$= \frac{3a}{4}\cos\theta + \frac{a}{4}\cos^3\theta - \frac{a}{4}\cos\theta\sin^2\theta - \frac{2a}{4}\sin^2\theta\cos\theta$
$= \frac{3a}{4}\cos\theta + \frac{a}{4}\cos^3\theta - \frac{3a}{4}(\cos\theta)(1 - \cos^2\theta) = a\cos^3\theta$;
$y = \left(a - \frac{a}{4}\right)\sin\theta - \frac{a}{4}\sin\left(\frac{a - \left(\frac{a}{4}\right)}{\left(\frac{a}{4}\right)}\theta\right) = \frac{3a}{4}\sin\theta - \frac{a}{4}\sin 3\theta = \frac{3a}{4}\sin\theta - \frac{a}{4}(\sin\theta\cos 2\theta + \cos\theta\sin 2\theta)$
$= \frac{3a}{4}\sin\theta - \frac{a}{4}((\sin\theta)(\cos^2\theta - \sin^2\theta) + (\cos\theta)(2\sin\theta\cos\theta))$
$= \frac{3a}{4}\sin\theta - \frac{a}{4}\sin\theta\cos^2\theta + \frac{a}{4}\sin^3\theta - \frac{2a}{4}\cos^2\theta\sin\theta$
$= \frac{3a}{4}\sin\theta - \frac{3a}{4}\sin\theta\cos^2\theta + \frac{a}{4}\sin^3\theta$
$= \frac{3a}{4}\sin\theta - \frac{3a}{4}(\sin\theta)(1 - \sin^2\theta) + \frac{a}{4}\sin^3\theta = a\sin^3\theta$.

37. Draw line AM in the figure and note that $\angle AMO$ is a right angle since it is an inscribed angle which spans the diameter of a circle. Then $AN^2 = MN^2 + AM^2$. Now, $OA = a$, $\frac{AN}{a} = \tan t$, and $\frac{AM}{a} = \sin t$. Next $MN = OP$ $\Rightarrow OP^2 = AN^2 - AM^2 = a^2\tan^2 t - a^2\sin^2 t$ $\Rightarrow OP = \sqrt{a^2\tan^2 t - a^2\sin^2 t}$ $= (a\sin t)\sqrt{\sec^2 t - 1} = \frac{a\sin^2 t}{\cos t}$. In triangle BPO, $x = OP\sin t = \frac{a\sin^3 t}{\cos t} = a\sin^2 t\tan t$ and $y = OP\cos t = a\sin^2 t \Rightarrow x = a\sin^2 t\tan t$ and $y = a\sin^2 t$.



38. Let the x-axis be the line the wheel rolls along with the y-axis through a low point of the trochoid (see the accompanying figure).

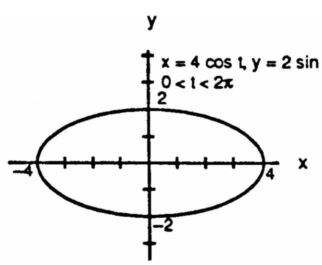


Let θ denote the angle through which the wheel turns. Then $h = a\theta$ and $k = a$. Next introduce $x'y'$-axes parallel to the xy-axes and having their origin at the center C of the wheel. Then $x' = b\cos\alpha$ and $y' = b\sin\alpha$, where $\alpha = \frac{3\pi}{2} - \theta$. It follows that $x' = b\cos\left(\frac{3\pi}{2} - \theta\right) = -b\sin\theta$ and $y' = b\sin\left(\frac{3\pi}{2} - \theta\right) = -b\cos\theta \Rightarrow x = h + x' = a\theta - b\sin\theta$ and $y = k + y' = a - b\cos\theta$ are parametric equations of the trochoid.

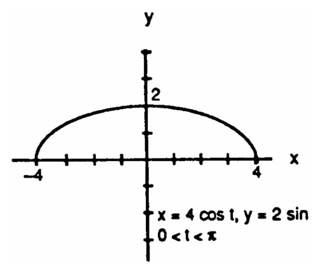
39. $D = \sqrt{(x-2)^2 + \left(y - \frac{1}{2}\right)^2} \Rightarrow D^2 = (x-2)^2 + \left(y - \frac{1}{2}\right)^2 = (t-2)^2 + \left(t^2 - \frac{1}{2}\right)^2 \Rightarrow D^2 = t^4 - 4t + \frac{17}{4}$ $\Rightarrow \frac{d(D^2)}{dt} = 4t^3 - 4 = 0 \Rightarrow t = 1$. The second derivative is always positive for $t \neq 0 \Rightarrow t = 1$ gives a local minimum for D^2 (and hence D) which is an absolute minimum since it is the only extremum $\Rightarrow$ the closest point on the parabola is $(1, 1)$.

40. $D = \sqrt{\left(2\cos t - \frac{3}{4}\right)^2 + (\sin t - 0)^2} \Rightarrow D^2 = \left(2\cos t - \frac{3}{4}\right)^2 + \sin^2 t \Rightarrow \frac{d(D^2)}{dt}$ $= 2\left(2\cos t - \frac{3}{4}\right)(-2\sin t) + 2\sin t\cos t = (-2\sin t)\left(3\cos t - \frac{3}{2}\right) = 0 \Rightarrow -2\sin t = 0$ or $3\cos t - \frac{3}{2} = 0$ $\Rightarrow t = 0, \pi$ or $t = \frac{\pi}{3}, \frac{5\pi}{3}$. Now $\frac{d^2(D^2)}{dt^2} = -6\cos^2 t + 3\cos t + 6\sin^2 t$ so that $\frac{d^2(D^2)}{dt^2}(0) = -3 \Rightarrow$ relative maximum, $\frac{d^2(D^2)}{dt^2}(\pi) = -9 \Rightarrow$ relative maximum, $\frac{d^2(D^2)}{dt^2}\left(\frac{\pi}{3}\right) = \frac{9}{2} \Rightarrow$ relative minimum, and $\frac{d^2(D^2)}{dt^2}\left(\frac{5\pi}{3}\right) = \frac{9}{2} \Rightarrow$ relative minimum. Therefore both $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ give points on the ellipse closest to the point $\left(\frac{3}{4}, 0\right) \Rightarrow \left(1, \frac{\sqrt{3}}{2}\right)$ and $\left(1, -\frac{\sqrt{3}}{2}\right)$ are the desired points.

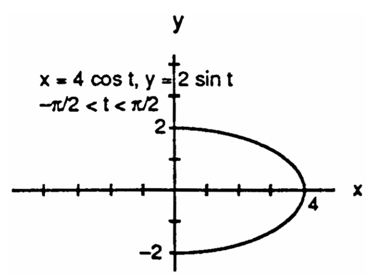
41. (a)



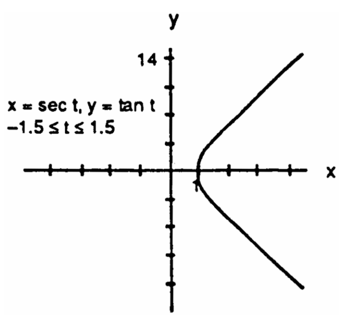
(b)



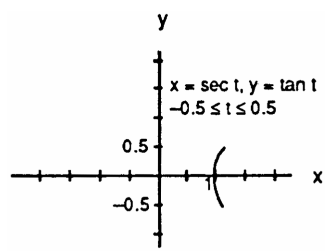
(c)



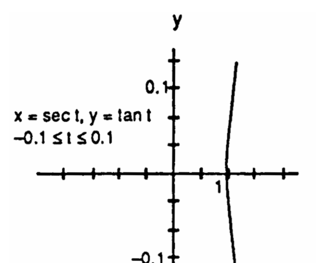
42. (a)



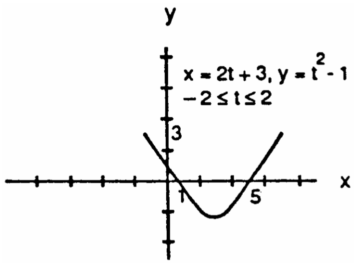
(b)



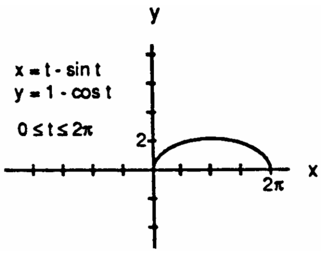
(c)



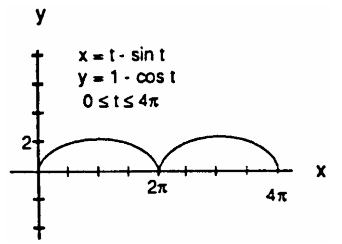
43.



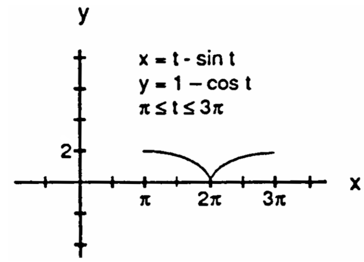
44. (a)



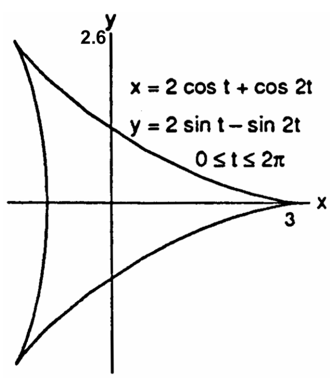
(b)



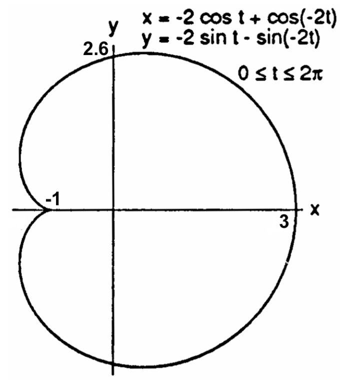
(c)



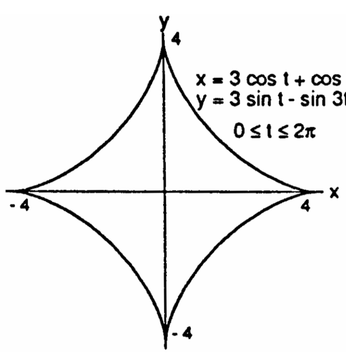
45. (a)



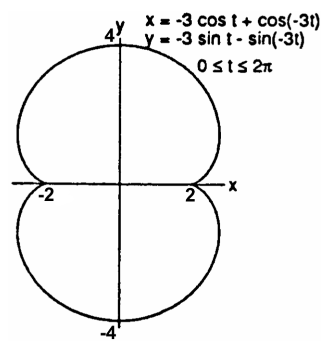
(b)



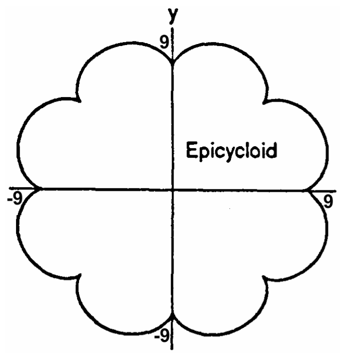
46. (a)



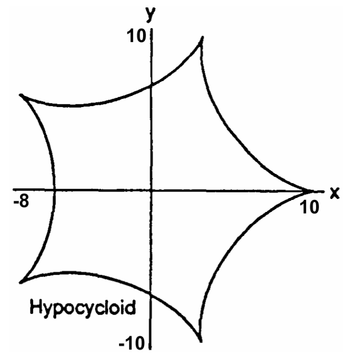
(b)



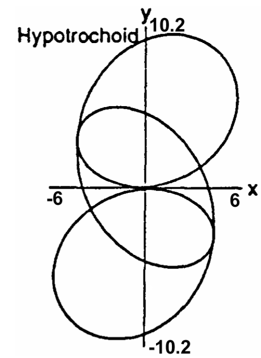
47. (a)



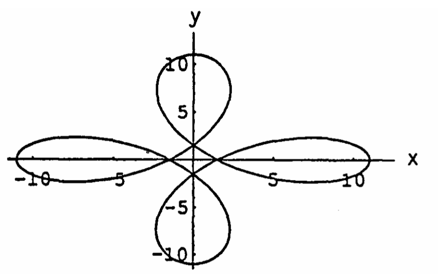
(b)



(c)

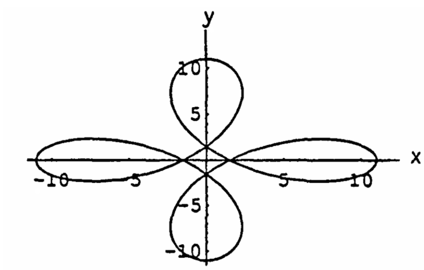


48. (a)



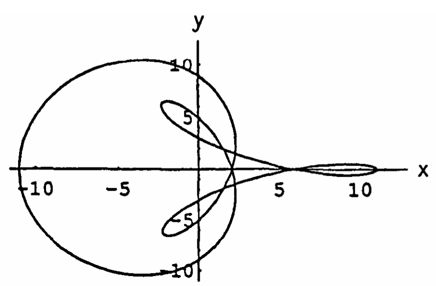
$x = 6\cos t + 5\cos 3t,\quad y = 6\sin t - 5\sin 3t,$
$0 \le t \le 2\pi$

(b)



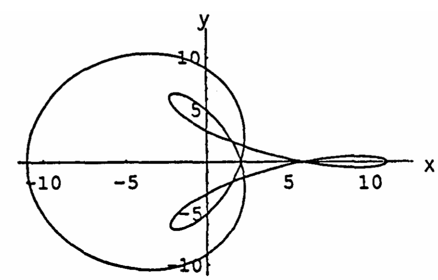
$x = 6\cos 2t + 5\cos 6t,\quad y = 6\sin 2t - 5\sin 6t,$
$0 \le t \le \pi$

(c)



$x = 6\cos t + 5\cos 3t,\quad y = 6\sin 2t - 5\sin 3t,$
$0 \le t \le 2\pi$

(d)



$x = 6\cos 2t + 5\cos 6t,\quad y = 6\sin 4t - 5\sin 6t,$
$0 \le t \le \pi$

11.2 CALCULUS WITH PARAMETRIC CURVES

1. $t = \frac{\pi}{4} \Rightarrow x = 2\cos\frac{\pi}{4} = \sqrt{2}$, $y = 2\sin\frac{\pi}{4} = \sqrt{2}$; $\frac{dx}{dt} = -2\sin t$, $\frac{dy}{dt} = 2\cos t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t} = -\cot t$ $\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1$; tangent line is $y - \sqrt{2} = -1\left(x - \sqrt{2}\right)$ or $y = -x + 2\sqrt{2}$; $\frac{dy'}{dt} = \csc^2 t$ $\Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\csc^2 t}{-2\sin t} = -\frac{1}{2\sin^3 t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = -\sqrt{2}$

2. $t = -\frac{1}{6} \Rightarrow x = \sin\left(2\pi\left(-\frac{1}{6}\right)\right) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, $y = \cos\left(2\pi\left(-\frac{1}{6}\right)\right) = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$; $\frac{dx}{dt} = 2\pi\cos 2\pi t$, $\frac{dy}{dt} = -2\pi\sin 2\pi t \Rightarrow \frac{dy}{dx} = \frac{-2\pi\sin 2\pi t}{2\pi\cos 2\pi t} = -\tan 2\pi t \Rightarrow \left.\frac{dy}{dx}\right|_{t=-\frac{1}{6}} = -\tan\left(2\pi\left(-\frac{1}{6}\right)\right) = -\tan\left(-\frac{\pi}{3}\right) = \sqrt{3}$; tangent line is $y - \frac{1}{2} = \sqrt{3}\left[x - \left(-\frac{\sqrt{3}}{2}\right)\right]$ or $y = \sqrt{3}x + 2$; $\frac{dy'}{dt} = -2\pi\sec^2 2\pi t \Rightarrow \frac{d^2y}{dx^2} = \frac{-2\pi\sec^2 2\pi t}{2\pi\cos 2\pi t}$ $= -\frac{1}{\cos^3 2\pi t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-\frac{1}{6}} = -8$

3. $t = \frac{\pi}{4} \Rightarrow x = 4 \sin \frac{\pi}{4} = 2\sqrt{2}$, $y = 2 \cos \frac{\pi}{4} = \sqrt{2}$; $\frac{dx}{dt} = 4 \cos t$, $\frac{dy}{dt} = -2 \sin t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2 \sin t}{4 \cos t}$
$= -\frac{1}{2} \tan t \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\frac{1}{2} \tan \frac{\pi}{4} = -\frac{1}{2}$; tangent line is $y - \sqrt{2} = -\frac{1}{2}\left(x - 2\sqrt{2}\right)$ or $y = -\frac{1}{2}x + 2\sqrt{2}$;
$\frac{dy'}{dt} = -\frac{1}{2} \sec^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\frac{1}{2}\sec^2 t}{4 \cos t} = -\frac{1}{8 \cos^3 t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = -\frac{\sqrt{2}}{4}$

4. $t = \frac{2\pi}{3} \Rightarrow x = \cos \frac{2\pi}{3} = -\frac{1}{2}$, $y = \sqrt{3} \cos \frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$; $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = -\sqrt{3} \sin t \Rightarrow \frac{dy}{dx} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \sqrt{3}$; tangent line is $y - \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}\left[x - \left(-\frac{1}{2}\right)\right]$ or $y = \sqrt{3}x$; $\frac{dy'}{dt} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{0}{-\sin t} = 0$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{2\pi}{3}} = 0$

5. $t = \frac{1}{4} \Rightarrow x = \frac{1}{4}$, $y = \frac{1}{2}$; $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = \frac{1}{2\sqrt{t}} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2\sqrt{t}} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{1}{4}} = \frac{1}{2\sqrt{\frac{1}{4}}} = 1$; tangent line is
$y - \frac{1}{2} = 1 \cdot \left(x - \frac{1}{4}\right)$ or $y = x + \frac{1}{4}$; $\frac{dy'}{dt} = -\frac{1}{4} t^{-3/2} \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = -\frac{1}{4} t^{-3/2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = -2$

6. $t = -\frac{\pi}{4} \Rightarrow x = \sec^2\left(-\frac{\pi}{4}\right) - 1 = 1$, $y = \tan\left(-\frac{\pi}{4}\right) = -1$; $\frac{dx}{dt} = 2 \sec^2 t \tan t$, $\frac{dy}{dt} = \sec^2 t$
$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 t}{2 \sec^2 t \tan t} = \frac{1}{2 \tan t} = \frac{1}{2} \cot t \Rightarrow \left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{4}} = \frac{1}{2} \cot\left(-\frac{\pi}{4}\right) = -\frac{1}{2}$; tangent line is
$y - (-1) = -\frac{1}{2}(x - 1)$ or $y = -\frac{1}{2}x - \frac{1}{2}$; $\frac{dy'}{dt} = -\frac{1}{2} \csc^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{1}{2}\csc^2 t}{2 \sec^2 t \tan t} = -\frac{1}{4} \cot^3 t$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}} = \frac{1}{4}$

7. $t = \frac{\pi}{6} \Rightarrow x = \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}$, $y = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$; $\frac{dx}{dt} = \sec t \tan t$, $\frac{dy}{dt} = \sec^2 t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
$= \frac{\sec^2 t}{\sec t \tan t} = \csc t \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \csc \frac{\pi}{6} = 2$; tangent line is $y - \frac{1}{\sqrt{3}} = 2\left(x - \frac{2}{\sqrt{3}}\right)$ or $y = 2x - \sqrt{3}$;
$\frac{dy'}{dt} = -\csc t \cot t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\csc t \cot t}{\sec t \tan t} = -\cot^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{6}} = -3\sqrt{3}$

8. $t = 3 \Rightarrow x = -\sqrt{3+1} = -2$, $y = \sqrt{3(3)} = 3$; $\frac{dx}{dt} = -\frac{1}{2}(t+1)^{-1/2}$, $\frac{dy}{dt} = \frac{3}{2}(3t)^{-1/2} \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{3}{2}\right)(3t)^{-1/2}}{\left(-\frac{1}{2}\right)(t+1)^{-1/2}}$
$= -\frac{3\sqrt{t+1}}{\sqrt{3t}} = \left.\frac{dy}{dx}\right|_{t=3} = \frac{-3\sqrt{3+1}}{\sqrt{3(3)}} = -2$; tangent line is $y - 3 = -2[x - (-2)]$ or $y = -2x - 1$;
$\frac{dy'}{dt} = \frac{\sqrt{3t}\left[-\frac{3}{2}(t+1)^{-1/2}\right] + 3\sqrt{t+1}\left[\frac{3}{2}(3t)^{-1/2}\right]}{3t} = \frac{3}{2t\sqrt{3t}\sqrt{t+1}} \Rightarrow \frac{d^2y}{dx^2} = \frac{\left(\frac{3}{2t\sqrt{3t}\sqrt{t+1}}\right)}{\left(\frac{-1}{2\sqrt{t+1}}\right)} = -\frac{3}{t\sqrt{3t}}$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=3} = -\frac{1}{3}$

9. $t = -1 \Rightarrow x = 5$, $y = 1$; $\frac{dx}{dt} = 4t$, $\frac{dy}{dt} = 4t^3 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{4t} = t^2 \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1} = (-1)^2 = 1$; tangent line is
$y - 1 = 1 \cdot (x - 5)$ or $y = x - 4$; $\frac{dy'}{dt} = 2t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{2t}{4t} = \frac{1}{2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-1} = \frac{1}{2}$

10. $t = 1 \Rightarrow x = 1$, $y = -2$; $\frac{dx}{dt} = -\frac{1}{t^2}$, $\frac{dy}{dt} = \frac{1}{t} \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{t}\right)}{\left(-\frac{1}{t^2}\right)} = -t \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = -1$; tangent line is
$y - (-2) = -1(x - 1)$ or $y = -x - 1$; $\frac{dy'}{dt} = -1 \Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{\left(-\frac{1}{t^2}\right)} = t^2 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=1} = 1$

11. $t = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3} - \sin \frac{\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$, $y = 1 - \cos \frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2}$; $\frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
$= \frac{\sin t}{1 - \cos t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}$; tangent line is $y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$

$\Rightarrow\ y = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + 2;\ \frac{dy'}{dt} = \frac{(1-\cos t)(\cos t) - (\sin t)(\sin t)}{(1-\cos t)^2} = \frac{-1}{1-\cos t} \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left(\frac{-1}{1-\cos t}\right)}{1-\cos t}$
$= \frac{-1}{(1-\cos t)^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = -4$

12. $t = \frac{\pi}{2} \Rightarrow x = \cos\frac{\pi}{2} = 0,\ y = 1 + \sin\frac{\pi}{2} = 2;\ \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t \Rightarrow \frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -\cot\frac{\pi}{2} = 0$; tangent line is $y = 2$; $\frac{dy'}{dt} = \csc^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{\csc^2 t}{-\sin t} = -\csc^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = -1$

13. $t = 2 \Rightarrow x = \frac{1}{2+1} = \frac{1}{3},\ y = \frac{2}{2-1} = 2;\ \frac{dx}{dt} = \frac{-1}{(t+1)^2},\ \frac{dy}{dt} = \frac{-1}{(t-1)^2} \Rightarrow \frac{dy}{dx} = \frac{(t+1)^2}{(t-1)^2} \Rightarrow \left.\frac{dy}{dx}\right|_{t=2} = \frac{(2+1)^2}{(2-1)^2} = 9;$
tangent line is $y = 9x - 1$; $\frac{dy'}{dt} = -\frac{4(t+1)}{(t-1)^3} \Rightarrow \frac{d^2y}{dx^2} = \frac{4(t+1)^3}{(t-1)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{4(2+1)^3}{(2-1)^3} = 108$

14. $t = 0 \Rightarrow x = 0 + e^0 = 1,\ y = 1 - e^0 = 0;\ \frac{dx}{dt} = 1 + e^t,\ \frac{dy}{dt} = -e^t \Rightarrow \frac{dy}{dx} = \frac{-e^t}{1+e^t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=0} = \frac{-e^0}{1+e^0} = -\frac{1}{2};$
tangent line is $y = -\frac{1}{2}x + \frac{1}{2}$; $\frac{dy'}{dt} = \frac{-e^t}{(1+e^t)^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{-e^t}{(1+e^t)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{-e^0}{(1+e^0)^3} = -\frac{1}{8}$

15. $x^3 + 2t^2 = 9 \Rightarrow 3x^2\frac{dx}{dt} + 4t = 0 \Rightarrow 3x^2\frac{dx}{dt} = -4t \Rightarrow \frac{dx}{dt} = \frac{-4t}{3x^2};$
$2y^3 - 3t^2 = 4 \Rightarrow 6y^2\frac{dy}{dt} - 6t = 0 \Rightarrow \frac{dy}{dt} = \frac{6t}{6y^2} = \frac{t}{y^2}$; thus $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{t}{y^2}\right)}{\left(\frac{-4t}{3x^2}\right)} = \frac{t(3x^2)}{y^2(-4t)} = \frac{3x^2}{-4y^2}$; $t = 2$
$\Rightarrow x^3 + 2(2)^2 = 9 \Rightarrow x^3 + 8 = 9 \Rightarrow x^3 = 1 \Rightarrow x = 1;\ t = 2 \Rightarrow 2y^3 - 3(2)^2 = 4$
$\Rightarrow 2y^3 = 16 \Rightarrow y^3 = 8 \Rightarrow y = 2$; therefore $\left.\frac{dy}{dx}\right|_{t=2} = \frac{3(1)^2}{-4(2)^2} = -\frac{3}{16}$

16. $x = \sqrt{5-\sqrt{t}} \Rightarrow \frac{dx}{dt} = \frac{1}{2}\left(5-\sqrt{t}\right)^{-1/2}\left(-\frac{1}{2}t^{-1/2}\right) = -\frac{1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}$; $y(t-1) = \sqrt{t} \Rightarrow y + (t-1)\frac{dy}{dt} = \frac{1}{2}t^{-1/2}$
$\Rightarrow (t-1)\frac{dy}{dt} = \frac{1}{2\sqrt{t}} - y \Rightarrow \frac{dy}{dt} = \frac{\frac{1}{2\sqrt{t}} - y}{(t-1)} = \frac{1-2y\sqrt{t}}{2t\sqrt{t}-2\sqrt{t}}$; thus $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1-2y\sqrt{t}}{2t\sqrt{t}-2\sqrt{t}}}{\frac{-1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}} = \frac{1-2y\sqrt{t}}{2\sqrt{t}(t-1)}\cdot\frac{4\sqrt{t}\sqrt{5-\sqrt{t}}}{-1}$
$= \frac{2\left(1-2y\sqrt{t}\right)\sqrt{5-\sqrt{t}}}{1-t}$; $t = 4 \Rightarrow x = \sqrt{5-\sqrt{4}} = \sqrt{3}$; $t = 4 \Rightarrow y\cdot 3 = \sqrt{4} \Rightarrow y = \frac{2}{3}$
therefore, $\left.\frac{dy}{dx}\right|_{t=4} = \frac{2\left(1-2\left(\frac{2}{3}\right)\sqrt{4}\right)\sqrt{5-\sqrt{4}}}{1-4} = \frac{10\sqrt{3}}{9}$

17. $x + 2x^{3/2} = t^2 + t \Rightarrow \frac{dx}{dt} + 3x^{1/2}\frac{dx}{dt} = 2t + 1 \Rightarrow \left(1 + 3x^{1/2}\right)\frac{dx}{dt} = 2t + 1 \Rightarrow \frac{dx}{dt} = \frac{2t+1}{1+3x^{1/2}}$; $y\sqrt{t+1} + 2t\sqrt{y} = 4$
$\Rightarrow \frac{dy}{dt}\sqrt{t+1} + y\left(\frac{1}{2}\right)(t+1)^{-1/2} + 2\sqrt{y} + 2t\left(\frac{1}{2}y^{-1/2}\right)\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt}\sqrt{t+1} + \frac{y}{2\sqrt{t+1}} + 2\sqrt{y} + \left(\frac{t}{\sqrt{y}}\right)\frac{dy}{dt} = 0$
$\Rightarrow \left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right)\frac{dy}{dt} = \frac{-y}{2\sqrt{t+1}} - 2\sqrt{y} \Rightarrow \frac{dy}{dt} = \frac{\left(\frac{-y}{2\sqrt{t+1}} - 2\sqrt{y}\right)}{\left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right)} = \frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}$; thus
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}\right)}{\left(\frac{2t+1}{1+3x^{1/2}}\right)}$; $t = 0 \Rightarrow x + 2x^{3/2} = 0 \Rightarrow x\left(1 + 2x^{1/2}\right) = 0 \Rightarrow x = 0$; $t = 0$
$\Rightarrow y\sqrt{0+1} + 2(0)\sqrt{y} = 4 \Rightarrow y = 4$; therefore $\left.\frac{dy}{dx}\right|_{t=0} = \frac{\left(\frac{-4\sqrt{4} - 4(4)\sqrt{0+1}}{2\sqrt{4}(0+1) + 2(0)\sqrt{0+1}}\right)}{\left(\frac{2(0)+1}{1+3(0)^{1/2}}\right)} = -6$

18. $x\sin t + 2x = t \Rightarrow \frac{dx}{dt}\sin t + x\cos t + 2\frac{dx}{dt} = 1 \Rightarrow (\sin t + 2)\frac{dx}{dt} = 1 - x\cos t \Rightarrow \frac{dx}{dt} = \frac{1 - x\cos t}{\sin t + 2};$
$t\sin t - 2t = y \Rightarrow \sin t + t\cos t - 2 = \frac{dy}{dt}$; thus $\frac{dy}{dx} = \frac{\sin t + t\cos t - 2}{\left(\frac{1 - x\cos t}{\sin t + 2}\right)}$; $t = \pi \Rightarrow x\sin\pi + 2x = \pi$
$\Rightarrow x = \frac{\pi}{2}$; therefore $\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{\sin\pi + \pi\cos\pi - 2}{\left[\frac{1 - \left(\frac{\pi}{2}\right)\cos\pi}{\sin\pi + 2}\right]} = \frac{-4\pi - 8}{2+\pi} = -4$

19. $x = t^3 + t,\ y + 2t^3 = 2x + t^2 \Rightarrow \frac{dx}{dt} = 3t^2 + 1,\ \frac{dy}{dt} + 6t^2 = 2\frac{dx}{dt} + 2t \Rightarrow \frac{dy}{dt} = 2(3t^2+1) + 2t - 6t^2 = 2t + 2$
$\Rightarrow \frac{dy}{dx} = \frac{2t+2}{3t^2+1} \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = \frac{2(1)+2}{3(1)^2+1} = 1$

20. $t = \ln(x - t),\ y = t e^t \Rightarrow 1 = \frac{1}{x-t}\left(\frac{dx}{dt} - 1\right) \Rightarrow x - t = \frac{dx}{dt} - 1 \Rightarrow \frac{dx}{dt} = x - t + 1,\ \frac{dy}{dt} = t e^t + e^t;$
$\Rightarrow \frac{dy}{dx} = \frac{t e^t + e^t}{x - t + 1};\ t = 0 \Rightarrow 0 = \ln(x - 0) \Rightarrow x = 1 \Rightarrow \left.\frac{dy}{dx}\right|_{t=0} = \frac{(0)e^0 + e^0}{1 - 0 + 1} = \frac{1}{2}$

21. $A = \int_0^{2\pi} y\,dx = \int_0^{2\pi} a(1-\cos t)a(1-\cos t)dt = a^2\int_0^{2\pi}(1-\cos t)^2 dt = a^2\int_0^{2\pi}(1 - 2\cos t + \cos^2 t)dt$
$= a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right)dt = a^2\int_0^{2\pi}\left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right)dt = a^2\left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$
$= a^2(3\pi - 0 + 0) - 0 = 3\pi\,a^2$

22. $A = \int_0^1 x\,dy = \int_0^1 (t - t^2)(-e^{-t})dt \quad \left[u = t - t^2 \Rightarrow du = (1-2t)dt;\ dv = (-e^{-t})dt \Rightarrow v = e^{-t}\right]$
$= e^{-t}(t - t^2)\Big|_0^1 - \int_0^1 e^{-t}(1-2t)dt \quad \left[u = 1 - 2t \Rightarrow du = -2dt;\ dv = e^{-t}dt \Rightarrow v = -e^{-t}\right]$
$= e^{-t}(t - t^2)\Big|_0^1 - \left[-e^{-t}(1-2t)\Big|_0^1 - \int_0^1 2e^{-t}dt\right] = \left[e^{-t}(t - t^2) + e^{-t}(1-2t) - 2e^{-t}\right]\Big|_0^1$
$= (e^{-1}(0) + e^{-1}(-1) - 2e^{-1}) - (e^0(0) + e^0(1) - 2e^0) = 1 - 3e^{-1} = 1 - \frac{3}{e}$

23. $A = 2\int_\pi^0 y\,dx = 2\int_\pi^0 (b\sin t)(-a\sin t)dt = 2ab\int_0^\pi \sin^2 t\,dt = 2ab\int_0^\pi \frac{1-\cos 2t}{2}\,dt = ab\int_0^\pi (1 - \cos 2t)\,dt$
$= ab\left[t - \frac{1}{2}\sin 2t\right]_0^\pi = ab((\pi - 0) - 0) = \pi\,ab$

24. (a) $x = t^2,\ y = t^6,\ 0 \le t \le 1 \Rightarrow A = \int_0^1 y\,dx = \int_0^1 (t^6)2t\,dt = \int_0^1 2t^7\,dt = \left[\frac{1}{4}t^8\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$
(b) $x = t^3,\ y = t^9,\ 0 \le t \le 1 \Rightarrow A = \int_0^1 y\,dx = \int_0^1 (t^9)3t^2\,dt = \int_0^1 3t^{11}\,dt = \left[\frac{1}{4}t^{12}\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$

25. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 1 + \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (1+\cos t)^2} = \sqrt{2 + 2\cos t}$
$\Rightarrow$ Length $= \int_0^\pi \sqrt{2 + 2\cos t}\,dt = \sqrt{2}\int_0^\pi \sqrt{\left(\frac{1-\cos t}{1-\cos t}\right)(1+\cos t)}\,dt = \sqrt{2}\int_0^\pi \sqrt{\frac{\sin^2 t}{1-\cos t}}\,dt$
$= \sqrt{2}\int_0^\pi \frac{\sin t}{\sqrt{1-\cos t}}\,dt$ (since $\sin t \ge 0$ on $[0,\pi]$); $[u = 1 - \cos t \Rightarrow du = \sin t\,dt;\ t = 0 \Rightarrow u = 0,$
$t = \pi \Rightarrow u = 2] \to \sqrt{2}\int_0^2 u^{-1/2}\,du = \sqrt{2}\left[2u^{1/2}\right]_0^2 = 4$

26. $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 3t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^2)^2 + (3t)^2} = \sqrt{9t^4 + 9t^2} = 3t\sqrt{t^2+1}$ (since $t \ge 0$ on $\left[0, \sqrt{3}\right]$)
$\Rightarrow$ Length $= \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\,dt;\ \left[u = t^2 + 1 \Rightarrow \frac{3}{2}\,du = 3t\,dt;\ t = 0 \Rightarrow u = 1,\ t = \sqrt{3} \Rightarrow u = 4\right]$
$\to \int_1^4 \frac{3}{2}u^{1/2}\,du = \left[u^{3/2}\right]_1^4 = (8 - 1) = 7$

27. $\frac{dx}{dt} = t$ and $\frac{dy}{dt} = (2t+1)^{1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2 + (2t+1)} = \sqrt{(t+1)^2} = |t+1| = t + 1$ since $0 \le t \le 4$
$\Rightarrow$ Length $= \int_0^4 (t+1)\,dt = \left[\frac{t^2}{2} + t\right]_0^4 = (8 + 4) = 12$

28. $\frac{dx}{dt} = (2t+3)^{1/2}$ and $\frac{dy}{dt} = 1+t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t+3)+(1+t)^2} = \sqrt{t^2+4t+4} = |t+2| = t+2$ since $0 \le t \le 3 \Rightarrow$ Length $= \int_0^3 (t+2)\,dt = \left[\frac{t^2}{2}+2t\right]_0^3 = \frac{21}{2}$

29. $\frac{dx}{dt} = 8t\cos t$ and $\frac{dy}{dt} = 8t\sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t\cos t)^2+(8t\sin t)^2} = \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t}$ $= |8t| = 8t$ since $0 \le t \le \frac{\pi}{2} \Rightarrow$ Length $= \int_0^{\pi/2} 8t\,dt = \left[4t^2\right]_0^{\pi/2} = \pi^2$

30. $\frac{dx}{dt} = \left(\frac{1}{\sec t+\tan t}\right)(\sec t\tan t + \sec^2 t) - \cos t = \sec t - \cos t$ and $\frac{dy}{dt} = -\sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ $= \sqrt{(\sec t-\cos t)^2 + (-\sin t)^2} = \sqrt{\sec^2 t - 1} = \sqrt{\tan^2 t} = |\tan t| = \tan t$ since $0 \le t \le \frac{\pi}{3}$ $\Rightarrow$ Length $= \int_0^{\pi/3} \tan t\,dt = \int_0^{\pi/3} \frac{\sin t}{\cos t}\,dt = \left[-\ln|\cos t|\right]_0^{\pi/3} = -\ln\frac{1}{2} + \ln 1 = \ln 2$

31. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2+(\cos t)^2} = 1 \Rightarrow$ Area $= \int 2\pi y\,ds$ $= \int_0^{2\pi} 2\pi(2+\sin t)(1)dt = 2\pi\left[2t-\cos t\right]_0^{2\pi} = 2\pi[(4\pi-1)-(0-1)] = 8\pi^2$

32. $\frac{dx}{dt} = t^{1/2}$ and $\frac{dy}{dt} = t^{-1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t+t^{-1}} = \sqrt{\frac{t^2+1}{t}} \Rightarrow$ Area $= \int 2\pi x\,ds$ $= \int_0^{\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{\frac{t^2+1}{t}}\,dt = \frac{4\pi}{3}\int_0^{\sqrt{3}} t\sqrt{t^2+1}\,dt$; $[u = t^2+1 \Rightarrow du = 2t\,dt;\ t=0 \Rightarrow u=1,$ $\left.t=\sqrt{3} \Rightarrow u=4\right] \rightarrow \int_1^4 \frac{2\pi}{3}\sqrt{u}\,du = \left[\frac{4\pi}{9}u^{3/2}\right]_1^4 = \frac{28\pi}{9}$

Note: $\int_0^{\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{\frac{t^2+1}{t}}\,dt$ is an improper integral but $\lim\limits_{t\to 0^+} f(t)$ exists and is equal to 0, where $f(t) = 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{\frac{t^2+1}{t}}$. Thus the discontinuity is removable: define $F(t) = f(t)$ for $t>0$ and $F(0)=0$ $\Rightarrow \int_0^{\sqrt{3}} F(t)\,dt = \frac{28\pi}{9}$.

33. $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = t+\sqrt{2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1^2+\left(t+\sqrt{2}\right)^2} = \sqrt{t^2+2\sqrt{2}t+3} \Rightarrow$ Area $= \int 2\pi x\,ds$ $= \int_{-\sqrt{2}}^{\sqrt{2}} 2\pi\left(t+\sqrt{2}\right)\sqrt{t^2+2\sqrt{2}t+3}\,dt$; $\left[u = t^2+2\sqrt{2}t+3 \Rightarrow du = \left(2t+2\sqrt{2}\right)dt;\ t=-\sqrt{2} \Rightarrow u=1,\right.$ $\left.t=\sqrt{2} \Rightarrow u=9\right] \rightarrow \int_1^9 \pi\sqrt{u}\,du = \left[\frac{2}{3}\pi u^{3/2}\right]_1^9 = \frac{2\pi}{3}(27-1) = \frac{52\pi}{3}$

34. From Exercise 30, $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \tan t \Rightarrow$ Area $= \int 2\pi y\,ds = \int_0^{\pi/3} 2\pi\cos t\tan t\,dt = 2\pi\int_0^{\pi/3}\sin t\,dt$ $= 2\pi\left[-\cos t\right]_0^{\pi/3} = 2\pi\left[-\frac{1}{2}-(-1)\right] = \pi$

35. $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2^2+1^2} = \sqrt{5} \Rightarrow$ Area $= \int 2\pi y\,ds = \int_0^1 2\pi(t+1)\sqrt{5}\,dt$ $= 2\pi\sqrt{5}\left[\frac{t^2}{2}+t\right]_0^1 = 3\pi\sqrt{5}$. Check: slant height is $\sqrt{5} \Rightarrow$ Area is $\pi(1+2)\sqrt{5} = 3\pi\sqrt{5}$.

36. $\frac{dx}{dt} = h$ and $\frac{dy}{dt} = r \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{h^2 + r^2} \Rightarrow \text{Area} = \int 2\pi y\, ds = \int_0^1 2\pi rt\sqrt{h^2 + r^2}\, dt$
$= 2\pi r\sqrt{h^2 + r^2} \int_0^1 t\, dt = 2\pi r\sqrt{h^2 + r^2} \left[\frac{t^2}{2}\right]_0^1 = \pi r\sqrt{h^2 + r^2}$. Check: slant height is $\sqrt{h^2 + r^2} \Rightarrow$ Area is $\pi r\sqrt{h^2 + r^2}$.

37. Let the density be $\delta = 1$. Then $x = \cos t + t \sin t \Rightarrow \frac{dx}{dt} = t \cos t$, and $y = \sin t - t \cos t \Rightarrow \frac{dy}{dt} = t \sin t$
$\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(t \cos t)^2 + (t \sin t)^2} = |t|\, dt = t\, dt$ since $0 \le t \le \frac{\pi}{2}$. The curve's mass is $M = \int dm = \int_0^{\pi/2} t\, dt = \frac{\pi^2}{8}$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\pi/2} (\sin t - t \cos t)\, t\, dt = \int_0^{\pi/2} t \sin t\, dt - \int_0^{\pi/2} t^2 \cos t\, dt$
$= [\sin t - t \cos t]_0^{\pi/2} - [t^2 \sin t - 2 \sin t + 2t \cos t]_0^{\pi/2} = 3 - \frac{\pi^2}{4}$, where we integrated by parts. Therefore,
$\overline{y} = \frac{M_x}{M} = \frac{\left(3 - \frac{\pi^2}{4}\right)}{\left(\frac{\pi^2}{8}\right)} = \frac{24}{\pi^2} - 2$. Next, $M_y = \int \tilde{x}\, dm = \int_0^{\pi/2} (\cos t + t \sin t)\, t\, dt = \int_0^{\pi/2} t \cos t\, dt + \int_0^{\pi/2} t^2 \sin t\, dt$
$= [\cos t + t \sin t]_0^{\pi/2} + [-t^2 \cos t + 2 \cos t + 2t \sin t]_0^{\pi/2} = \frac{3\pi}{2} - 3$, again integrating by parts. Hence
$\overline{x} = \frac{M_y}{M} = \frac{\left(\frac{3\pi}{2} - 3\right)}{\left(\frac{\pi^2}{8}\right)} = \frac{12}{\pi} - \frac{24}{\pi^2}$. Therefore $(\overline{x}, \overline{y}) = \left(\frac{12}{\pi} - \frac{24}{\pi^2}, \frac{24}{\pi^2} - 2\right)$.

38. Let the density be $\delta = 1$. Then $x = e^t \cos t \Rightarrow \frac{dx}{dt} = e^t \cos t - e^t \sin t$, and $y = e^t \sin t \Rightarrow \frac{dy}{dt} = e^t \sin t + e^t \cos t$
$\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2}\, dt = \sqrt{2e^{2t}}\, dt = \sqrt{2}\, e^t\, dt$.
The curve's mass is $M = \int dm = \int_0^{\pi} \sqrt{2}\, e^t\, dt = \sqrt{2}\, e^{\pi} - \sqrt{2}$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\pi} (e^t \sin t)\left(\sqrt{2}\, e^t\right) dt$
$= \int_0^{\pi} \sqrt{2}\, e^{2t} \sin t\, dt = \sqrt{2}\left[\frac{e^{2t}}{5}(2 \sin t - \cos t)\right]_0^{\pi} = \sqrt{2}\left(\frac{e^{2\pi}}{5} + \frac{1}{5}\right) \Rightarrow \overline{y} = \frac{M_x}{M} = \frac{\sqrt{2}\left(\frac{e^{2\pi}}{5} + \frac{1}{5}\right)}{\sqrt{2}\, e^{\pi} - \sqrt{2}} = \frac{e^{2\pi} + 1}{5(e^{\pi} - 1)}$.
Next $M_y = \int \tilde{x}\, dm = \int_0^{\pi} (e^t \cos t)\left(\sqrt{2}\, e^t\right) dt = \int_0^{\pi} \sqrt{2}\, e^{2t} \cos t\, dt = \sqrt{2}\left[\frac{e^{2t}}{5}(2 \cos t + \sin t)\right]_0^{\pi} = -\sqrt{2}\left(\frac{2e^{2\pi}}{5} + \frac{2}{5}\right)$
$\Rightarrow \overline{x} = \frac{M_y}{M} = \frac{-\sqrt{2}\left(\frac{2e^{2\pi}}{5} + \frac{2}{5}\right)}{\sqrt{2}\, e^{\pi} - \sqrt{2}} = -\frac{2e^{2\pi} + 2}{5(e^{\pi} - 1)}$. Therefore $(\overline{x}, \overline{y}) = \left(-\frac{2e^{2\pi} + 2}{5(e^{\pi} - 1)}, \frac{e^{2\pi} + 1}{5(e^{\pi} - 1)}\right)$.

39. Let the density be $\delta = 1$. Then $x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$, and $y = t + \sin t \Rightarrow \frac{dy}{dt} = 1 + \cos t$
$\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(-\sin t)^2 + (1 + \cos t)^2}\, dt = \sqrt{2 + 2 \cos t}\, dt$. The curve's mass is $M = \int dm = \int_0^{\pi} \sqrt{2 + 2 \cos t}\, dt = \sqrt{2} \int_0^{\pi} \sqrt{1 + \cos t}\, dt = \sqrt{2} \int_0^{\pi} \sqrt{2 \cos^2\left(\frac{t}{2}\right)}\, dt = 2 \int_0^{\pi} \left|\cos\left(\frac{t}{2}\right)\right| dt$
$= 2 \int_0^{\pi} \cos\left(\frac{t}{2}\right) dt$ $\left(\text{since } 0 \le t \le \pi \Rightarrow 0 \le \frac{t}{2} \le \frac{\pi}{2}\right) = 2\left[2 \sin\left(\frac{t}{2}\right)\right]_0^{\pi} = 4$. Also $M_x = \int \tilde{y}\, dm$
$= \int_0^{\pi} (t + \sin t)\left(2 \cos \frac{t}{2}\right) dt = \int_0^{\pi} 2t \cos\left(\frac{t}{2}\right) dt + \int_0^{\pi} 2 \sin t \cos\left(\frac{t}{2}\right) dt$
$= 2\left[4 \cos\left(\frac{t}{2}\right) + 2t \sin\left(\frac{t}{2}\right)\right]_0^{\pi} + 2\left[-\frac{1}{3} \cos\left(\frac{3}{2}t\right) - \cos\left(\frac{1}{2}t\right)\right]_0^{\pi} = 4\pi - \frac{16}{3} \Rightarrow \overline{y} = \frac{M_x}{M} = \frac{\left(4\pi - \frac{16}{3}\right)}{4} = \pi - \frac{4}{3}$.
Next $M_y = \int \tilde{x}\, dm = \int_0^{\pi} (\cos t)\left(2 \cos \frac{t}{2}\right) dt = \int_0^{\pi} \cos t \cos\left(\frac{t}{2}\right) dt = 2\left[\sin\left(\frac{t}{2}\right) + \frac{\sin\left(\frac{3}{2}t\right)}{3}\right]_0^{\pi} = 2 - \frac{2}{3}$
$= \frac{4}{3} \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{\left(\frac{4}{3}\right)}{4} = \frac{1}{3}$. Therefore $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \pi - \frac{4}{3}\right)$.

40. Let the density be $\delta = 1$. Then $x = t^3 \Rightarrow \frac{dx}{dt} = 3t^2$, and $y = \frac{3t^2}{2} \Rightarrow \frac{dy}{dt} = 3t \Rightarrow dm = 1 \cdot ds$
$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\left(3t^2\right)^2 + (3t)^2}\, dt = 3\,|t|\,\sqrt{t^2 + 1}\, dt = 3t\sqrt{t^2 + 1}\, dt$ since $0 \le t \le \sqrt{3}$. The curve's mass is $M = \int dm = \int_0^{\sqrt{3}} 3t\sqrt{t^2 + 1}\, dt = \left[\left(t^2 + 1\right)^{3/2}\right]_0^{\sqrt{3}} = 7$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\sqrt{3}} \frac{3t^2}{2}\left(3t\sqrt{t^2 + 1}\right) dt$
$= \frac{9}{2} \int_0^{\sqrt{3}} t^3 \sqrt{t^2 + 1}\, dt = \frac{87}{5} = 17.4$ (by computer) $\Rightarrow \overline{y} = \frac{M_x}{M} = \frac{17.4}{7} \approx 2.49$. Next $M_y = \int \tilde{x}\, dm$

$= \int_0^{\sqrt{3}} t^3 \cdot 3t\,\sqrt{(t^2+1)}\,dt = 3\int_0^{\sqrt{3}} t^4\sqrt{t^2+1}\,dt \approx 16.4849$ (by computer) $\Rightarrow\ \overline{x} = \frac{M_y}{M} = \frac{16.4849}{7} \approx 2.35$.
Therefore, $(\overline{x}, \overline{y}) \approx (2.35, 2.49)$.

41. (a) $\frac{dx}{dt} = -2\sin 2t$ and $\frac{dy}{dt} = 2\cos 2t \ \Rightarrow\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2} = 2$

$\Rightarrow$ Length $= \int_0^{\pi/2} 2\,dt = [2t]_0^{\pi/2} = \pi$

(b) $\frac{dx}{dt} = \pi\cos\pi t$ and $\frac{dy}{dt} = -\pi\sin\pi t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\pi\cos\pi t)^2 + (-\pi\sin\pi t)^2} = \pi$

$\Rightarrow$ Length $= \int_{-1/2}^{1/2} \pi\,dt = [\pi t]_{-1/2}^{1/2} = \pi$

42. (a) $x = g(y)$ has the parametrization $x = g(y)$ and $y = y$ for $c \le y \le d \ \Rightarrow\ \frac{dx}{dy} = g'(y)$ and $\frac{dy}{dy} = 1$; then

Length $= \int_c^d \sqrt{\left(\frac{dy}{dy}\right)^2 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy$

(b) $x = y^{3/2},\ 0 \le y \le \frac{4}{3} \Rightarrow \frac{dx}{dy} = \frac{3}{2}y^{1/2} \Rightarrow L = \int_0^{4/3} \sqrt{1 + \left(\frac{3}{2}y^{1/2}\right)^2}\,dy = \int_0^{4/3} \sqrt{1 + \frac{9}{4}y}\,dy = \left[\frac{4}{9}\cdot\frac{2}{3}\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^{4/3}$

$= \frac{8}{27}(4)^{3/2} - \frac{8}{27}(1)^{3/2} = \frac{56}{27}$

(c) $x = \frac{3}{2}y^{2/3},\ 0 \le y \le 1 \Rightarrow \frac{dx}{dy} = y^{-1/3} \Rightarrow L = \int_0^1 \sqrt{1 + \left(y^{-1/3}\right)^2}\,dy = \int_0^1 \sqrt{1 + \frac{1}{y^{2/3}}}\,dy = \lim_{a\to 0^+} \int_a^1 \sqrt{\frac{y^{2/3}+1}{y^{2/3}}}\,dy$

$= \lim_{a\to 0^+} \frac{3}{2}\int_a^1 \left(y^{2/3}+1\right)^{1/2}\left(\frac{2}{3}y^{-1/3}\right)dy = \lim_{a\to 0^+} \left[\frac{3}{2}\cdot\frac{2}{3}\left(y^{2/3}+1\right)^{3/2}\right]_a^1 = \lim_{a\to 0^+}\left((2)^{3/2} - \left(a^{2/3}+1\right)^{3/2}\right) = 2\sqrt{2} - 1$

43. $x = (1 + 2\sin\theta)\cos\theta,\ y = (1 + 2\sin\theta)\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos^2\theta - \sin\theta(1 + 2\sin\theta),\ \frac{dy}{d\theta} = 2\cos\theta\sin\theta + \cos\theta(1 + 2\sin\theta)$
$\Rightarrow \frac{dy}{dx} = \frac{2\cos\theta\sin\theta + \cos\theta(1+2\sin\theta)}{2\cos^2\theta - \sin\theta(1+2\sin\theta)} = \frac{4\cos\theta\sin\theta + \cos\theta}{2\cos^2\theta - 2\sin^2\theta - \sin\theta} = \frac{2\sin 2\theta + \cos\theta}{2\cos 2\theta - \sin\theta}$

(a) $x = (1 + 2\sin(0))\cos(0) = 1,\ y = (1 + 2\sin(0))\sin(0) = 0;\ \left.\frac{dy}{dx}\right|_{\theta=0} = \frac{2\sin(2(0)) + \cos(0)}{2\cos(2(0)) - \sin(0)} = \frac{0+1}{2-0} = \frac{1}{2}$

(b) $x = \left(1 + 2\sin\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right) = 0,\ y = \left(1 + 2\sin\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right) = 3;\ \left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{2\sin\left(2\left(\frac{\pi}{2}\right)\right) + \cos\left(\frac{\pi}{2}\right)}{2\cos\left(2\left(\frac{\pi}{2}\right)\right) - \sin\left(\frac{\pi}{2}\right)} = \frac{0+0}{-2-1} = 0$

(c) $x = \left(1 + 2\sin\left(\frac{4\pi}{3}\right)\right)\cos\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}-1}{2},\ y = \left(1 + 2\sin\left(\frac{4\pi}{3}\right)\right)\sin\left(\frac{4\pi}{3}\right) = \frac{3-\sqrt{3}}{2};\ \left.\frac{dy}{dx}\right|_{\theta=4\pi/3} = \frac{2\sin\left(2\left(\frac{4\pi}{3}\right)\right) + \cos\left(\frac{4\pi}{3}\right)}{2\cos\left(2\left(\frac{4\pi}{3}\right)\right) - \sin\left(\frac{4\pi}{3}\right)}$

$= \frac{\sqrt{3} - \frac{1}{2}}{-1 + \frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}-1}{\sqrt{3}-2} = -\left(4 + 3\sqrt{3}\right)$

44. $x = t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi \Rightarrow \frac{dx}{dt} = 1,\ \frac{dy}{dt} = \sin t \Rightarrow \frac{dy}{dx} = \frac{\sin t}{1} = \sin t \Rightarrow \frac{d}{dt}\left(\frac{dy}{dx}\right) = \cos t \Rightarrow \frac{d^2y}{dx^2} = \frac{\cos t}{1} = \cos t$. The maximum and minimum slope will occur at points that maximize/minimize $\frac{dy}{dx}$, in other words, points where $\frac{d^2y}{dx^2} = 0$
$\Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2} \Rightarrow \frac{d^2y}{dx^2} = +++ \underset{\pi/2}{|} - - - \underset{3\pi/2}{|} +++$

(a) the maximum slope is $\left.\frac{dy}{dx}\right|_{t=\pi/2} = \sin\left(\frac{\pi}{2}\right) = 1$, which occurs at $x = \frac{\pi}{2},\ y = 1 - \cos\left(\frac{\pi}{2}\right) = 1$

(a) the minimum slope is $\left.\frac{dy}{dx}\right|_{t=3\pi/2} = \sin\left(\frac{3\pi}{2}\right) = -1$, which occurs at $x = \frac{3\pi}{2},\ y = 1 - \cos\left(\frac{3\pi}{2}\right) = 1$

45. $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = 2\cos 2t \ \Rightarrow\ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos 2t}{\cos t} = \frac{2\left(2\cos^2 t - 1\right)}{\cos t}$; then $\frac{dy}{dx} = 0 \ \Rightarrow\ \frac{2\left(2\cos^2 t - 1\right)}{\cos t} = 0$
$\Rightarrow\ 2\cos^2 t - 1 = 0 \ \Rightarrow\ \cos t = \pm\frac{1}{\sqrt{2}} \ \Rightarrow\ t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. In the 1st quadrant: $t = \frac{\pi}{4} \ \Rightarrow\ x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $y = \sin 2\left(\frac{\pi}{4}\right) = 1 \ \Rightarrow\ \left(\frac{\sqrt{2}}{2}, 1\right)$ is the point where the tangent line is horizontal. At the origin: $x = 0$ and $y = 0$

$\Rightarrow \sin t = 0 \Rightarrow t = 0$ or $t = \pi$ and $\sin 2t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$; thus $t = 0$ and $t = \pi$ give the tangent lines at the origin. Tangents at origin: $\left.\frac{dy}{dx}\right|_{t=0} = 2 \Rightarrow y = 2x$ and $\left.\frac{dy}{dx}\right|_{t=\pi} = -2 \Rightarrow y = -2x$

46. $\frac{dx}{dt} = 2\cos 2t$ and $\frac{dy}{dt} = 3\cos 3t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos 3t}{2\cos 2t} = \frac{3(\cos 2t \cos t - \sin 2t \sin t)}{2(2\cos^2 t - 1)}$
$= \frac{3[(2\cos^2 t - 1)(\cos t) - 2\sin t \cos t \sin t]}{2(2\cos^2 t - 1)} = \frac{(3\cos t)(2\cos^2 t - 1 - 2\sin^2 t)}{2(2\cos^2 t - 1)} = \frac{(3\cos t)(4\cos^2 t - 3)}{2(2\cos^2 t - 1)}$; then
$\frac{dy}{dx} = 0 \Rightarrow \frac{(3\cos t)(4\cos^2 t - 3)}{2(2\cos^2 t - 1)} = 0 \Rightarrow 3\cos t = 0$ or $4\cos^2 t - 3 = 0$: $3\cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$ and
$4\cos^2 t - 3 = 0 \Rightarrow \cos t = \pm\frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. In the 1st quadrant: $t = \frac{\pi}{6} \Rightarrow x = \sin 2\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $y = \sin 3\left(\frac{\pi}{6}\right) = 1 \Rightarrow \left(\frac{\sqrt{3}}{2}, 1\right)$ is the point where the graph has a horizontal tangent. At the origin: $x = 0$ and $y = 0 \Rightarrow \sin 2t = 0$ and $\sin 3t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \Rightarrow t = 0$ and $t = \pi$ give the tangent lines at the origin. Tangents at the origin: $\left.\frac{dy}{dx}\right|_{t=0} = \frac{3\cos 0}{2\cos 0} = \frac{3}{2} \Rightarrow y = \frac{3}{2}x$, and $\left.\frac{dy}{dx}\right|_{t=\pi}$
$= \frac{3\cos(3\pi)}{2\cos(2\pi)} = -\frac{3}{2} \Rightarrow y = -\frac{3}{2}x$

47. (a) $x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \le t \le 2\pi \Rightarrow \frac{dx}{dt} = a(1 - \cos t),\ \frac{dy}{dt} = a\sin t \Rightarrow$ Length
$$= \int_0^{2\pi} \sqrt{(a(1-\cos t))^2 + (a\sin t)^2}\, dt = \int_0^{2\pi} \sqrt{a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t}\, dt$$
$$= a\sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos t}\, dt = a\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\left(\tfrac{t}{2}\right)}\, dt = 2a\int_0^{2\pi} \sin\left(\tfrac{t}{2}\right) dt = \left[-4a\cos\left(\tfrac{t}{2}\right)\right]_0^{2\pi}$$
$$= -4a\cos\pi + 4a\cos(0) = 8a$$
(b) $a = 1 \Rightarrow x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi \Rightarrow \frac{dx}{dt} = 1 - \cos t,\ \frac{dy}{dt} = \sin t \Rightarrow$ Surface area $=$
$$= \int_0^{2\pi} 2\pi(1-\cos t)\sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt = \int_0^{2\pi} 2\pi(1-\cos t)\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$
$$= 2\pi\int_0^{2\pi}(1-\cos t)\sqrt{2 - 2\cos t}\, dt = 2\sqrt{2}\pi\int_0^{2\pi}(1-\cos t)^{3/2}\, dt = 2\sqrt{2}\pi\int_0^{2\pi}\left(1 - \cos\left(2\cdot\tfrac{t}{2}\right)\right)^{3/2} dt$$
$$= 2\sqrt{2}\pi\int_0^{2\pi}\left(2\sin^2\left(\tfrac{t}{2}\right)\right)^{3/2} dt = 8\pi\int_0^{2\pi}\sin^3\left(\tfrac{t}{2}\right) dt$$
$$\left[u = \tfrac{t}{2} \Rightarrow du = \tfrac{1}{2}dt \Rightarrow dt = 2\,du;\ t = 0 \Rightarrow u = 0,\ t = 2\pi \Rightarrow u = \pi\right]$$
$$= 16\pi\int_0^{\pi}\sin^3 u\, du = 16\pi\int_0^{\pi}\sin^2 u \sin u\, du = 16\pi\int_0^{\pi}(1 - \cos^2 u)\sin u\, du = 16\pi\int_0^{\pi}\sin u\, du - 16\pi\int_0^{\pi}\cos^2 u \sin u\, du$$
$$= \left[-16\pi\cos u + \tfrac{16\pi}{3}\cos^3 u\right]_0^{\pi} = \left(16\pi - \tfrac{16\pi}{3}\right) - \left(-16\pi + \tfrac{16\pi}{3}\right) = \tfrac{64\pi}{3}$$

48. $x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$; Volume $= \int_0^{2\pi} \pi\, y^2 dx = \int_0^{2\pi} \pi(1-\cos t)^2(1-\cos t)dt$
$$= \pi\int_0^{2\pi}(1 - 3\cos t + 3\cos^2 t - \cos^3 t)dt = \pi\int_0^{2\pi}\left(1 - 3\cos t + 3\left(\tfrac{1+\cos 2t}{2}\right) - \cos^2 t\cos t\right)dt$$
$$= \pi\int_0^{2\pi}\left(\tfrac{5}{2} - 3\cos t + \tfrac{3}{2}\cos 2t - (1 - \sin^2 t)\cos t\right)dt = \pi\int_0^{2\pi}\left(\tfrac{5}{2} - 4\cos t + \tfrac{3}{2}\cos 2t + \sin^2 t\cos t\right)dt$$
$$= \pi\left[\tfrac{5}{2}t - 4\sin t + \tfrac{3}{4}\sin 2t + \tfrac{1}{3}\sin^3 t\right]_0^{2\pi} = \pi(5\pi - 0 + 0 + 0) - 0 = 5\pi^2$$

47-50. Example CAS commands:

Maple:

```
with( plots );
with( student );
x := t -> t^3/3;
y := t -> t^2/2;
a := 0;
b := 1;
N := [2, 4, 8 ];
for n in N do
```

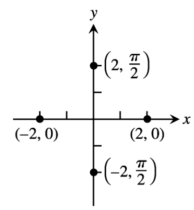
```
  tt := [seq( a+i*(b-a)/n, i=0..n )];
  pts := [seq([x(t),y(t)],t=tt)];
  L := simplify(add( student[distance](pts[i+1],pts[i]), i=1..n ));     # (b)
  T := sprintf("#47(a) (Section 11.2)\nn=%3d  L=%8.5f\n", n, L );
  P[n] := plot( [[x(t),y(t),t=a..b],pts], title=T ):                     # (a)
end do:
display( [seq(P[n],n=N)], insequence=true );
ds := t ->sqrt( simplify(D(x)(t)^2 + D(y)(t)^2) ):                       # (c)
L := Int( ds(t), t=a..b ):
L = evalf(L);
```

11.3 POLAR COORDINATES

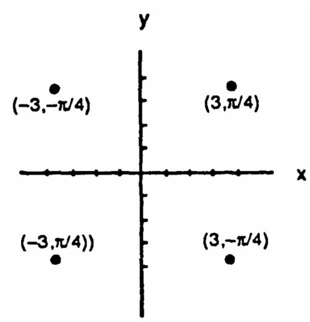
1. a, e; b, g; c, h; d, f

2. a, f; b, h; c, g; d, e

3. (a) $\left(2, \frac{\pi}{2} + 2n\pi\right)$ and $\left(-2, \frac{\pi}{2} + (2n+1)\pi\right)$, n an integer
   (b) $(2, 2n\pi)$ and $(-2, (2n+1)\pi)$, n an integer
   (c) $\left(2, \frac{3\pi}{2} + 2n\pi\right)$ and $\left(-2, \frac{3\pi}{2} + (2n+1)\pi\right)$, n an integer
   (d) $(2, (2n+1)\pi)$ and $(-2, 2n\pi)$, n an integer



4. (a) $\left(3, \frac{\pi}{4} + 2n\pi\right)$ and $\left(-3, \frac{5\pi}{4} + 2n\pi\right)$, n an integer
   (b) $\left(-3, \frac{\pi}{4} + 2n\pi\right)$ and $\left(3, \frac{5\pi}{4} + 2n\pi\right)$, n an integer
   (c) $\left(3, -\frac{\pi}{4} + 2n\pi\right)$ and $\left(-3, \frac{3\pi}{4} + 2n\pi\right)$, n an integer
   (d) $\left(-3, -\frac{\pi}{4} + 2n\pi\right)$ and $\left(3, \frac{3\pi}{4} + 2n\pi\right)$, n an integer



5. (a) $x = r\cos\theta = 3\cos 0 = 3,\ y = r\sin\theta = 3\sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(3, 0)$
   (b) $x = r\cos\theta = -3\cos 0 = -3,\ y = r\sin\theta = -3\sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(-3, 0)$
   (c) $x = r\cos\theta = 2\cos\frac{2\pi}{3} = -1,\ y = r\sin\theta = 2\sin\frac{2\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(-1, \sqrt{3}\right)$
   (d) $x = r\cos\theta = 2\cos\frac{7\pi}{3} = 1,\ y = r\sin\theta = 2\sin\frac{7\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(1, \sqrt{3}\right)$
   (e) $x = r\cos\theta = -3\cos\pi = 3,\ y = r\sin\theta = -3\sin\pi = 0 \Rightarrow$ Cartesian coordinates are $(3, 0)$
   (f) $x = r\cos\theta = 2\cos\frac{\pi}{3} = 1,\ y = r\sin\theta = 2\sin\frac{\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(1, \sqrt{3}\right)$
   (g) $x = r\cos\theta = -3\cos 2\pi = -3,\ y = r\sin\theta = -3\sin 2\pi = 0 \Rightarrow$ Cartesian coordinates are $(-3, 0)$
   (h) $x = r\cos\theta = -2\cos\left(-\frac{\pi}{3}\right) = -1,\ y = r\sin\theta = -2\sin\left(-\frac{\pi}{3}\right) = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(-1, \sqrt{3}\right)$

6. (a) $x = \sqrt{2}\cos\frac{\pi}{4} = 1,\ y = \sqrt{2}\sin\frac{\pi}{4} = 1 \Rightarrow$ Cartesian coordinates are $(1, 1)$
   (b) $x = 1\cos 0 = 1,\ y = 1\sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(1, 0)$
   (c) $x = 0\cos\frac{\pi}{2} = 0,\ y = 0\sin\frac{\pi}{2} = 0 \Rightarrow$ Cartesian coordinates are $(0, 0)$
   (d) $x = -\sqrt{2}\cos\left(\frac{\pi}{4}\right) = -1,\ y = -\sqrt{2}\sin\left(\frac{\pi}{4}\right) = -1 \Rightarrow$ Cartesian coordinates are $(-1, -1)$
   (e) $x = -3\cos\frac{5\pi}{6} = \frac{3\sqrt{3}}{2},\ y = -3\sin\frac{5\pi}{6} = -\frac{3}{2} \Rightarrow$ Cartesian coordinates are $\left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$
   (f) $x = 5\cos\left(\tan^{-1}\frac{4}{3}\right) = 3,\ y = 5\sin\left(\tan^{-1}\frac{4}{3}\right) = 4 \Rightarrow$ Cartesian coordinates are $(3, 4)$

(g) $x = -1 \cos 7\pi = 1$, $y = -1 \sin 7\pi = 0 \;\Rightarrow\;$ Cartesian coordinates are $(1, 0)$

(h) $x = 2\sqrt{3} \cos \frac{2\pi}{3} = -\sqrt{3}$, $y = 2\sqrt{3} \sin \frac{2\pi}{3} = 3 \;\Rightarrow\;$ Cartesian coordinates are $\left(-\sqrt{3}, 3\right)$

7. (a) $(1, 1) \Rightarrow r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\sin\theta = \frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \Rightarrow$ Polar coordinates are $\left(\sqrt{2}, \frac{\pi}{4}\right)$

(b) $(-3, 0) \Rightarrow r = \sqrt{(-3)^2 + 0^2} = 3$, $\sin\theta = 0$ and $\cos\theta = -1 \Rightarrow \theta = \pi \Rightarrow$ Polar coordinates are $(3, \pi)$

(c) $\left(\sqrt{3}, -1\right) \Rightarrow r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$, $\sin\theta = -\frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{11\pi}{6} \Rightarrow$ Polar coordinates are $\left(2, \frac{11\pi}{6}\right)$

(d) $(-3, 4) \Rightarrow r = \sqrt{(-3)^2 + 4^2} = 5$, $\sin\theta = \frac{4}{5}$ and $\cos\theta = -\frac{3}{5} \Rightarrow \theta = \pi - \arctan\left(\frac{4}{3}\right) \Rightarrow$ Polar coordinates are $\left(5, \pi - \arctan\left(\frac{4}{3}\right)\right)$

8. (a) $(-2, -2) \Rightarrow r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$, $\sin\theta = -\frac{1}{\sqrt{2}}$ and $\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = -\frac{3\pi}{4} \Rightarrow$ Polar coordinates are $\left(2\sqrt{2}, -\frac{3\pi}{4}\right)$

(b) $(0, 3) \Rightarrow r = \sqrt{0^2 + 3^2} = 3$, $\sin\theta = 1$ and $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ Polar coordinates are $\left(3, \frac{\pi}{2}\right)$

(c) $\left(-\sqrt{3}, 1\right) \Rightarrow r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$, $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{5\pi}{6} \Rightarrow$ Polar coordinates are $\left(2, \frac{5\pi}{6}\right)$

(d) $(5, -12) \Rightarrow r = \sqrt{5^2 + (-12)^2} = 13$, $\sin\theta = -\frac{12}{13}$ and $\cos\theta = \frac{5}{12} \Rightarrow \theta = -\arctan\left(\frac{12}{5}\right) \Rightarrow$ Polar coordinates are $\left(13, -\arctan\left(\frac{12}{5}\right)\right)$

9. (a) $(3, 3) \Rightarrow r = -\sqrt{3^2 + 3^2} = -3\sqrt{2}$, $\sin\theta = -\frac{1}{\sqrt{2}}$ and $\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{5\pi}{4} \Rightarrow$ Polar coordinates are $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$

(b) $(-1, 0) \Rightarrow r = -\sqrt{(-1)^2 + 0^2} = -1$, $\sin\theta = 0$ and $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow$ Polar coordinates are $(-1, 0)$

(c) $\left(-1, \sqrt{3}\right) \Rightarrow r = -\sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = -2$, $\sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{5\pi}{3} \Rightarrow$ Polar coordinates are $\left(-2, \frac{5\pi}{3}\right)$

(d) $(4, -3) \Rightarrow r = -\sqrt{4^2 + (-3)^2} = -5$, $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5} \Rightarrow \theta = \pi - \arctan\left(\frac{3}{4}\right) \Rightarrow$ Polar coordinates are $\left(-5, \pi - \arctan\left(\frac{4}{3}\right)\right)$

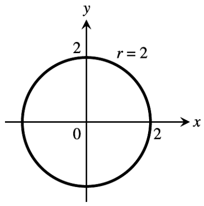
10. (a) $(-2, 0) \Rightarrow r = -\sqrt{(-2)^2 + 0^2} = -2$, $\sin\theta = 0$ and $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow$ Polar coordinates are $(-2, 0)$

(b) $(1, 0) \Rightarrow r = -\sqrt{1^2 + 0^2} = -1$, $\sin\theta = 0$ and $\cos\theta = -1 \Rightarrow \theta = \pi$ or $\theta = -\pi \Rightarrow$ Polar coordinates are $(-1, \pi)$ or $(-1, -\pi)$

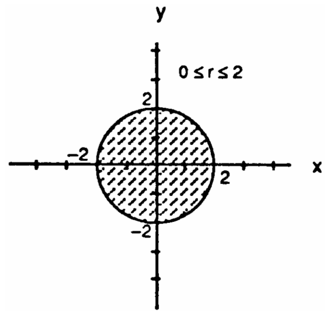
(c) $(0, -3) \Rightarrow r = -\sqrt{0^2 + (-3)^2} = -3$, $\sin\theta = 1$ and $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ Polar coordinates are $\left(-3, \frac{\pi}{2}\right)$

(d) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \Rightarrow r = -\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = -1$, $\sin\theta = -\frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{7\pi}{6}$ or $\theta = -\frac{5\pi}{6} \Rightarrow$ Polar coordinates are $\left(-1, \frac{7\pi}{6}\right)$ or $\left(-1, -\frac{5\pi}{6}\right)$

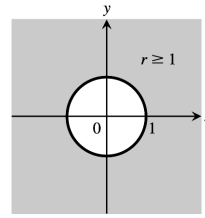
11.



12.



13.



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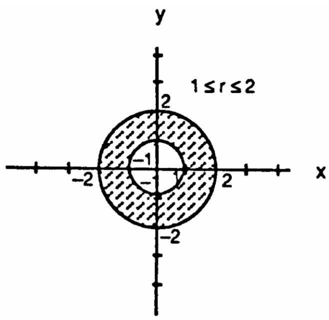

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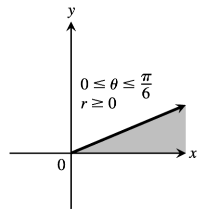

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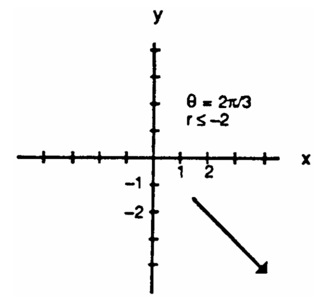

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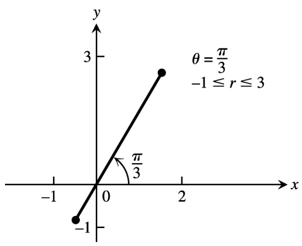

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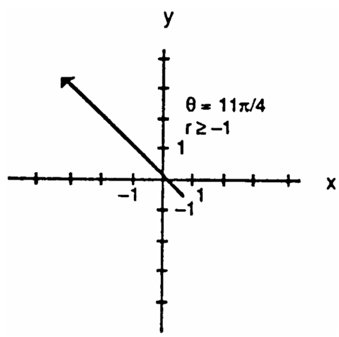

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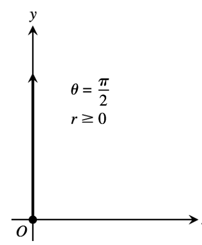

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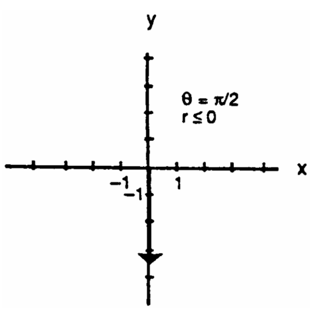

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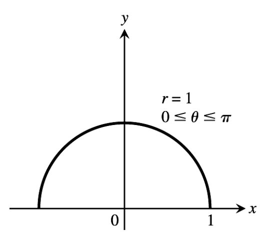

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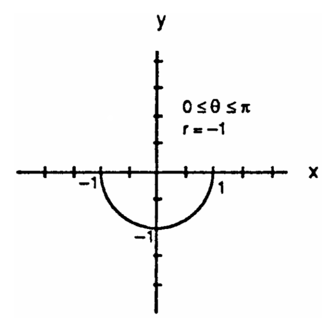

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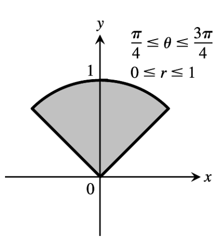

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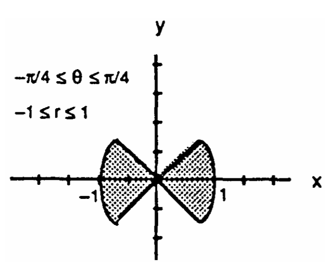

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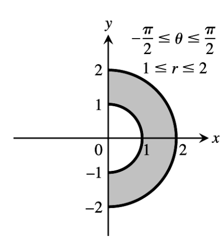
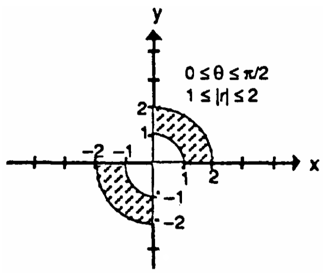


26. 

27. $r \cos\theta = 2 \Rightarrow x = 2$, vertical line through $(2, 0)$

28. $r \sin\theta = -1 \Rightarrow y = -1$, horizontal line through $(0, -1)$

29. $r \sin\theta = 0 \Rightarrow y = 0$, the x-axis

30. $r \cos\theta = 0 \Rightarrow x = 0$, the y-axis

31. $r = 4 \csc\theta \Rightarrow r = \frac{4}{\sin\theta} \Rightarrow r \sin\theta = 4 \Rightarrow y = 4$, a horizontal line through $(0, 4)$

32. $r = -3 \sec\theta \Rightarrow r = \frac{-3}{\cos\theta} \Rightarrow r \cos\theta = -3 \Rightarrow x = -3$, a vertical line through $(-3, 0)$

33. $r \cos\theta + r \sin\theta = 1 \Rightarrow x + y = 1$, line with slope $m = -1$ and intercept $b = 1$

34. $r \sin\theta = r \cos\theta \Rightarrow y = x$, line with slope $m = 1$ and intercept $b = 0$

35. $r^2 = 1 \Rightarrow x^2 + y^2 = 1$, circle with center $C = (0, 0)$ and radius 1

36. $r^2 = 4r \sin\theta \Rightarrow x^2 + y^2 = 4y \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow x^2 + (y - 2)^2 = 4$, circle with center $C = (0, 2)$ and radius 2

37. $r = \frac{5}{\sin\theta - 2\cos\theta} \Rightarrow r \sin\theta - 2r \cos\theta = 5 \Rightarrow y - 2x = 5$, line with slope $m = 2$ and intercept $b = 5$

38. $r^2 \sin 2\theta = 2 \Rightarrow 2r^2 \sin\theta \cos\theta = 2 \Rightarrow (r \sin\theta)(r \cos\theta) = 1 \Rightarrow xy = 1$, hyperbola with focal axis $y = x$

39. $r = \cot\theta \csc\theta = \left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{1}{\sin\theta}\right) \Rightarrow r \sin^2\theta = \cos\theta \Rightarrow r^2 \sin^2\theta = r \cos\theta \Rightarrow y^2 = x$, parabola with vertex $(0, 0)$ which opens to the right

40. $r = 4 \tan\theta \sec\theta \Rightarrow r = 4\left(\frac{\sin\theta}{\cos^2\theta}\right) \Rightarrow r \cos^2\theta = 4 \sin\theta \Rightarrow r^2 \cos^2\theta = 4r \sin\theta \Rightarrow x^2 = 4y$, parabola with vertex $= (0, 0)$ which opens upward

41. $r = (\csc\theta)\, e^{r\cos\theta} \Rightarrow r \sin\theta = e^{r\cos\theta} \Rightarrow y = e^x$, graph of the natural exponential function

42. $r \sin\theta = \ln r + \ln \cos\theta = \ln(r \cos\theta) \Rightarrow y = \ln x$, graph of the natural logarithm function

43. $r^2 + 2r^2 \cos\theta \sin\theta = 1 \Rightarrow x^2 + y^2 + 2xy = 1 \Rightarrow x^2 + 2xy + y^2 = 1 \Rightarrow (x + y)^2 = 1 \Rightarrow x + y = \pm 1$, two parallel straight lines of slope -1 and y-intercepts $b = \pm 1$

44. $\cos^2\theta = \sin^2\theta \Rightarrow r^2 \cos^2\theta = r^2 \sin^2\theta \Rightarrow x^2 = y^2 \Rightarrow |x| = |y| \Rightarrow \pm x = y$, two perpendicular lines through the origin with slopes 1 and -1, respectively.

45. $r^2 = -4r \cos\theta \Rightarrow x^2 + y^2 = -4x \Rightarrow x^2 + 4x + y^2 = 0 \Rightarrow x^2 + 4x + 4 + y^2 = 4 \Rightarrow (x + 2)^2 + y^2 = 4$, a circle with center $C(-2, 0)$ and radius 2

46. $r^2 = -6r\sin\theta \Rightarrow x^2 + y^2 = -6y \Rightarrow x^2 + y^2 + 6y = 0 \Rightarrow x^2 + y^2 + 6y + 9 = 9 \Rightarrow x^2 + (y+3)^2 = 9$, a circle with center $C(0, -3)$ and radius 3

47. $r = 8\sin\theta \Rightarrow r^2 = 8r\sin\theta \Rightarrow x^2 + y^2 = 8y \Rightarrow x^2 + y^2 - 8y = 0 \Rightarrow x^2 + y^2 - 8y + 16 = 16 \Rightarrow x^2 + (y-4)^2 = 16$, a circle with center $C(0, 4)$ and radius 4

48. $r = 3\cos\theta \Rightarrow r^2 = 3r\cos\theta \Rightarrow x^2 + y^2 = 3x \Rightarrow x^2 + y^2 - 3x = 0 \Rightarrow x^2 - 3x + \frac{9}{4} + y^2 = \frac{9}{4}$
$\Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$, a circle with center $C\left(\frac{3}{2}, 0\right)$ and radius $\frac{3}{2}$

49. $r = 2\cos\theta + 2\sin\theta \Rightarrow r^2 = 2r\cos\theta + 2r\sin\theta \Rightarrow x^2 + y^2 = 2x + 2y \Rightarrow x^2 - 2x + y^2 - 2y = 0$
$\Rightarrow (x-1)^2 + (y-1)^2 = 2$, a circle with center $C(1, 1)$ and radius $\sqrt{2}$

50. $r = 2\cos\theta - \sin\theta \Rightarrow r^2 = 2r\cos\theta - r\sin\theta \Rightarrow x^2 + y^2 = 2x - y \Rightarrow x^2 - 2x + y^2 + y = 0$
$\Rightarrow (x-1)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{5}{4}$, a circle with center $C\left(1, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$

51. $r\sin\left(\theta + \frac{\pi}{6}\right) = 2 \Rightarrow r\left(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right) = 2 \Rightarrow \frac{\sqrt{3}}{2}r\sin\theta + \frac{1}{2}r\cos\theta = 2 \Rightarrow \frac{\sqrt{3}}{2}y + \frac{1}{2}x = 2$
$\Rightarrow \sqrt{3}\,y + x = 4$, line with slope $m = -\frac{1}{\sqrt{3}}$ and intercept $b = \frac{4}{\sqrt{3}}$

52. $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5 \Rightarrow r\left(\sin\frac{2\pi}{3}\cos\theta - \cos\frac{2\pi}{3}\sin\theta\right) = 5 \Rightarrow \frac{\sqrt{3}}{2}r\cos\theta + \frac{1}{2}r\sin\theta = 5 \Rightarrow \frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5$
$\Rightarrow \sqrt{3}\,x + y = 10$, line with slope $m = -\sqrt{3}$ and intercept $b = 10$

53. $x = 7 \Rightarrow r\cos\theta = 7$

54. $y = 1 \Rightarrow r\sin\theta = 1$

55. $x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$

56. $x - y = 3 \Rightarrow r\cos\theta - r\sin\theta = 3$

57. $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$ or $r = -2$

58. $x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1 \Rightarrow r^2\left(\cos^2\theta - \sin^2\theta\right) = 1 \Rightarrow r^2\cos 2\theta = 1$

59. $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow 4x^2 + 9y^2 = 36 \Rightarrow 4r^2\cos^2\theta + 9r^2\sin^2\theta = 36$

60. $xy = 2 \Rightarrow (r\cos\theta)(r\sin\theta) = 2 \Rightarrow r^2\cos\theta\sin\theta = 2 \Rightarrow 2r^2\cos\theta\sin\theta = 4 \Rightarrow r^2\sin 2\theta = 4$

61. $y^2 = 4x \Rightarrow r^2\sin^2\theta = 4r\cos\theta \Rightarrow r\sin^2\theta = 4\cos\theta$

62. $x^2 + xy + y^2 = 1 \Rightarrow x^2 + y^2 + xy = 1 \Rightarrow r^2 + r^2\sin\theta\cos\theta = 1 \Rightarrow r^2(1 + \sin\theta\cos\theta) = 1$

63. $x^2 + (y-2)^2 = 4 \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow x^2 + y^2 = 4y \Rightarrow r^2 = 4r\sin\theta \Rightarrow r = 4\sin\theta$

64. $(x-5)^2 + y^2 = 25 \Rightarrow x^2 - 10x + 25 + y^2 = 25 \Rightarrow x^2 + y^2 = 10x \Rightarrow r^2 = 10r\cos\theta \Rightarrow r = 10\cos\theta$

65. $(x-3)^2 + (y+1)^2 = 4 \Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1 = 4 \Rightarrow x^2 + y^2 = 6x - 2y - 6 \Rightarrow r^2 = 6r\cos\theta - 2r\sin\theta - 6$

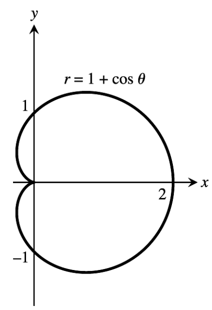
66. $(x+2)^2 + (y-5)^2 = 16 \Rightarrow x^2 + 4x + 4 + y^2 - 10y + 25 = 16 \Rightarrow x^2 + y^2 = -4x + 10y - 13$
$\Rightarrow r^2 = -4r\cos\theta + 10r\sin\theta - 13$

67. $(0, \theta)$ where θ is any angle

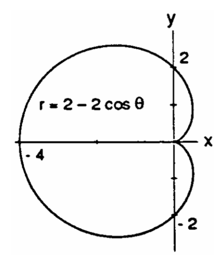
68. (a) $x = a \Rightarrow r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta} \Rightarrow r = a\sec\theta$
(b) $y = b \Rightarrow r\sin\theta = b \Rightarrow r = \frac{b}{\sin\theta} \Rightarrow r = b\csc\theta$

11.4 GRAPHING IN POLAR COORDINATES

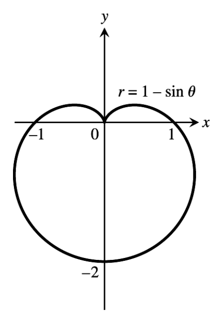
1. $1 + \cos(-\theta) = 1 + \cos\theta = r \Rightarrow$ symmetric about the x-axis; $1 + \cos(-\theta) \neq -r$ and $1 + \cos(\pi - \theta) = 1 - \cos\theta \neq r \Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



2. $2 - 2\cos(-\theta) = 2 - 2\cos\theta = r \Rightarrow$ symmetric about the x-axis; $2 - 2\cos(-\theta) \neq -r$ and $2 - 2\cos(\pi - \theta) = 2 + 2\cos\theta \neq r \Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



3. $1 - \sin(-\theta) = 1 + \sin\theta \neq r$ and $1 - \sin(\pi - \theta) = 1 - \sin\theta \neq -r \Rightarrow$ not symmetric about the x-axis; $1 - \sin(\pi - \theta) = 1 - \sin\theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



4. $1 + \sin(-\theta) = 1 - \sin\theta \neq r$ and $1 + \sin(\pi - \theta) = 1 + \sin\theta \neq -r \Rightarrow$ not symmetric about the x-axis; $1 + \sin(\pi - \theta) = 1 + \sin\theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin