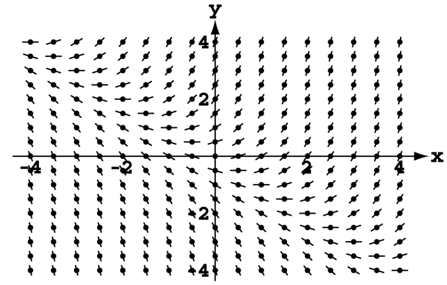
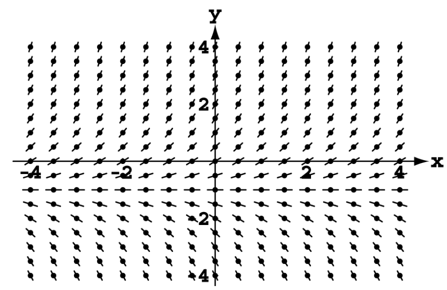
CHAPTER 9 FIRST-ORDER DIFFERENTIAL EQUATIONS

9.1 SOLUTIONS, SLOPE FIELDS AND EULER'S METHOD

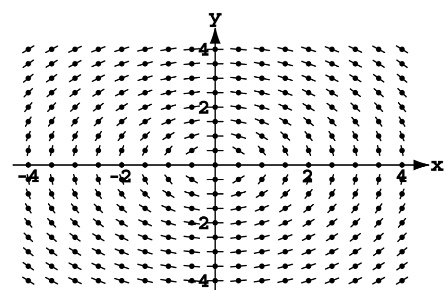
1. $y' = x + y \Rightarrow$ slope of 0 for the line $y = -x$.
   For $x, y > 0$, $y' = x + y \Rightarrow$ slope > 0 in Quadrant I.
   For $x, y < 0$, $y' = x + y \Rightarrow$ slope < 0 in Quadrant III.
   For $|y| > |x|$, $y > 0$, $x < 0$, $y' = x + y \Rightarrow$ slope > 0 in Quadrant II above $y = -x$.
   For $|y| < |x|$, $y > 0$, $x < 0$, $y' = x + y \Rightarrow$ slope < 0 in Quadrant II below $y = -x$.
   For $|y| < |x|$, $x > 0$, $y < 0$, $y' = x + y \Rightarrow$ slope > 0 in Quadrant IV above $y = -x$.
   For $|y| > |x|$, $x > 0$, $y < 0$, $y' = x + y \Rightarrow$ slope < 0 in Quadrant IV below $y = -x$.
   All of the conditions are seen in slope field (d).



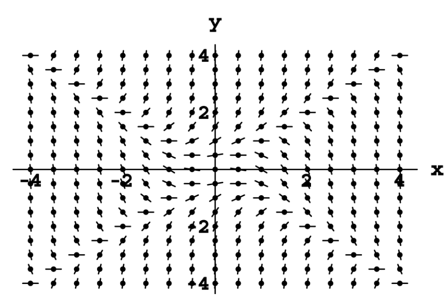
2. $y' = y + 1 \Rightarrow$ slope is constant for a given value of y, slope is 0 for $y = -1$, slope is positive for $y > 1$ and negative for $y < -1$. These characteristics are evident in slope field (c).



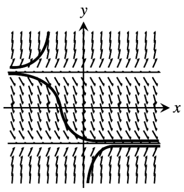
3. $y' = -\frac{x}{y} \Rightarrow$ slope $= 1$ on $y = -x$ and -1 on $y = x$.
   $y' = -\frac{x}{y} \Rightarrow$ slope $= 0$ on the y-axis, excluding $(0, 0)$, and is undefined on the x-axis. Slopes are positive for $x > 0, y < 0$ and $x < 0, y > 0$ (Quadrants II and IV), otherwise negative. Field (a) is consistent with these conditions.



4. $y' = y^2 - x^2 \Rightarrow$ slope is 0 for $y = x$ and for $y = -x$.
   For $|y| > |x|$ slope is positive and for $|y| < |x|$ slope is negative. Field (b) has these characteristics.



5.



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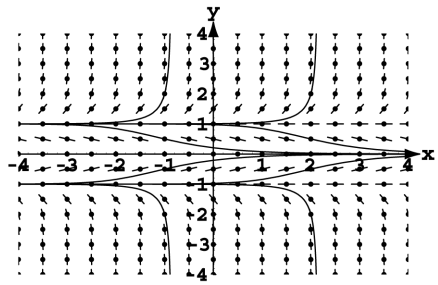


7. $y = -1 + \int_1^x (t - y(t))dt \Rightarrow \frac{dy}{dx} = x - y(x)$; $y(1) = -1 + \int_1^1 (t - y(t))dt = -1$; $\frac{dy}{dx} = x - y$, $y(1) = -1$

8. $y = \int_1^x \frac{1}{t} dt \Rightarrow \frac{dy}{dx} = \frac{1}{x}$; $y(1) = \int_1^1 \frac{1}{t} dt = 0$; $\frac{dy}{dx} = \frac{1}{x}$, $y(1) = 0$

9. $y = 2 - \int_0^x (1 + y(t)) \sin t \, dt \Rightarrow \frac{dy}{dx} = -(1 + y(x)) \sin x$; $y(0) = 2 - \int_0^0 (1 + y(t)) \sin t \, dt = 2$; $\frac{dy}{dx} = -(1 + y) \sin x$, $y(0) = 2$

10. $y = 1 + \int_0^x y(t) \, dt \Rightarrow \frac{dy}{dx} = y(x)$; $y(0) = 1 + \int_0^0 y(t) \, dt = 1$; $\frac{dy}{dx} = y$, $y(0) = 1$

11. $y_1 = y_0 + \left(1 - \frac{y_0}{x_0}\right) dx = -1 + \left(1 - \frac{-1}{2}\right)(.5) = -0.25,$

$y_2 = y_1 + \left(1 - \frac{y_1}{x_1}\right) dx = -0.25 + \left(1 - \frac{-0.25}{2.5}\right)(.5) = 0.3,$

$y_3 = y_2 + \left(1 - \frac{y_2}{x_2}\right) dx = 0.3 + \left(1 - \frac{0.3}{3}\right)(.5) = 0.75;$

$\frac{dy}{dx} + \left(\frac{1}{x}\right) y = 1 \Rightarrow P(x) = \frac{1}{x}, Q(x) = 1 \Rightarrow \int P(x)\, dx = \int \frac{1}{x}\, dx = \ln|x| = \ln x, x > 0 \Rightarrow v(x) = e^{\ln x} = x$

$\Rightarrow y = \frac{1}{x} \int x \cdot 1 \, dx = \frac{1}{x}\left(\frac{x^2}{2} + C\right); x = 2, y = -1 \Rightarrow -1 = 1 + \frac{C}{2} \Rightarrow C = -4 \Rightarrow y = \frac{x}{2} - \frac{4}{x}$

$\Rightarrow y(3.5) = \frac{3.5}{2} - \frac{4}{3.5} = \frac{4.25}{7} \approx 0.6071$

12. $y_1 = y_0 + x_0 (1 - y_0)\, dx = 0 + 1(1 - 0)(.2) = .2,$

$y_2 = y_1 + x_1 (1 - y_1)\, dx = .2 + 1.2(1 - .2)(.2) = .392,$

$y_3 = y_2 + x_2 (1 - y_2)\, dx = .392 + 1.4(1 - .392)(.2) = .5622;$

$\frac{dy}{1-y} = x\, dx \Rightarrow -\ln|1 - y| = \frac{x^2}{2} + C; x = 1, y = 0 \Rightarrow -\ln 1 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2} \Rightarrow \ln|1 - y| = -\frac{x^2}{2} + \frac{1}{2}$

$\Rightarrow y = 1 - e^{(1-x^2)/2} \Rightarrow y(1.6) \approx .5416$

13. $y_1 = y_0 + (2x_0y_0 + 2y_0)\, dx = 3 + [2(0)(3) + 2(3)](.2) = 4.2,$

$y_2 = y_1 + (2x_1y_1 + 2y_1)\, dx = 4.2 + [2(.2)(4.2) + 2(4.2)](.2) = 6.216,$

$y_3 = y_2 + (2x_2y_2 + 2y_2)\, dx = 6.216 + [2(.4)(6.216) + 2(6.216)](.2) = 9.6969;$

$\frac{dy}{dx} = 2y(x + 1) \Rightarrow \frac{dy}{y} = 2(x + 1)\, dx \Rightarrow \ln|y| = (x + 1)^2 + C; x = 0, y = 3 \Rightarrow \ln 3 = 1 + C \Rightarrow C = \ln 3 - 1$

$\Rightarrow \ln y = (x + 1)^2 + \ln 3 - 1 \Rightarrow y = e^{(x+1)^2 + \ln 3 - 1} = e^{\ln 3} e^{x^2 + 2x} = 3e^{x(x+2)} \Rightarrow y(.6) \approx 14.2765$

14. $y_1 = y_0 + y_0^2(1 + 2x_0)\, dx = 1 + 1^2[1 + 2(-1)](.5) = .5,$

$y_2 = y_1 + y_1^2(1 + 2x_1)\, dx = .5 + (.5)^2[1 + 2(-.5)](.5) = .5,$

$y_3 = y_2 + y_2^2(1 + 2x_2)\, dx = .5 + (.5)^2[1 + 2(0)](.5) = .625;$

$\frac{dy}{y^2} = (1 + 2x)\, dx \Rightarrow -\frac{1}{y} = x + x^2 + C; x = -1, y = 1 \Rightarrow -1 = -1 + (-1)^2 + C \Rightarrow C = -1 \Rightarrow \frac{1}{y} = 1 - x - x^2$

$\Rightarrow y = \frac{1}{1 - x - x^2} \Rightarrow y(.5) = \frac{1}{1 - .5 - (.5)^2} = 4$

15. $y_1 = y_0 + 2x_0 e^{x_0^2}\,dx = 2 + 2(0)(.1) = 2,$
$y_2 = y_1 + 2x_1 e^{x_1^2}\,dx = 2 + 2(.1)\,e^{.1^2}(.1) = 2.0202,$
$y_3 = y_2 + 2x_2 e^{x_2^2}\,dx = 2.0202 + 2(.2)\,e^{.2^2}(.1) = 2.0618,$
$dy = 2xe^{x^2}\,dx \;\Rightarrow\; y = e^{x^2} + C;\, y(0) = 2 \;\Rightarrow\; 2 = 1 + C \;\Rightarrow\; C = 1 \;\Rightarrow\; y = e^{x^2} + 1 \;\Rightarrow\; y(.3) = e^{.3^2} + 1 \approx 2.0942$

16. $y_1 = y_0 + \left(y_0\,e^{x_0}\right)dx = 2 + \left(2 \cdot e^0\right)(.5) = 3,$
$y_2 = y_1 + \left(y_1\,e^{x_1}\right)dx = 3 + \left(3 \cdot e^{0.5}\right)(.5) = 5.47308,$
$y_3 = y_2 + \left(y_2\,e^{x_2}\right)dx = 5.47308 + \left(5.47308 \cdot e^{1.0}\right)(.5) = 12.9118,$
$\frac{dy}{dx} = y\,e^x \;\Rightarrow\; \frac{dy}{y} = e^x\,dx \Rightarrow \ln|y| = e^x + C;\, x = 0,\, y = 2 \Rightarrow \ln 2 = 1 + C \Rightarrow C = \ln 2 - 1 \Rightarrow \ln|y| = e^x + \ln 2 - 1$
$\Rightarrow\; y = 2e^{e^x - 1} \Rightarrow y(1.5) = 2e^{e^{1.5} - 1} \approx 65.0292$

17. $y_1 = 1 + 1(.2) = 1.2,$
$y_2 = 1.2 + (1.2)(.2) = 1.44,$
$y_3 = 1.44 + (1.44)(.2) = 1.728,$
$y_4 = 1.728 + (1.728)(.2) = 2.0736,$
$y_5 = 2.0736 + (2.0736)(.2) = 2.48832;$
$\frac{dy}{y} = dx \;\Rightarrow\; \ln y = x + C_1 \;\Rightarrow\; y = Ce^x;\, y(0) = 1 \;\Rightarrow\; 1 = Ce^0 \;\Rightarrow\; C = 1 \;\Rightarrow\; y = e^x \;\Rightarrow\; y(1) = e \approx 2.7183$

18. $y_1 = 2 + \left(\frac{2}{1}\right)(.2) = 2.4,$
$y_2 = 2.4 + \left(\frac{2.4}{1.2}\right)(.2) = 2.8,$
$y_3 = 2.8 + \left(\frac{2.8}{1.4}\right)(.2) = 3.2,$
$y_4 = 3.2 + \left(\frac{3.2}{1.6}\right)(.2) = 3.6,$
$y_5 = 3.6 + \left(\frac{3.6}{1.8}\right)(.2) = 4;$
$\frac{dy}{y} = \frac{dx}{x} \;\Rightarrow\; \ln y = \ln x + C \;\Rightarrow\; y = kx;\, y(1) = 2 \;\Rightarrow\; 2 = k \;\Rightarrow\; y = 2x \;\Rightarrow\; y(2) = 4$

19. $y_1 = -1 + \left[\frac{(-1)^2}{\sqrt{1}}\right](.5) = -.5,$
$y_2 = -.5 + \left[\frac{(-.5)^2}{\sqrt{1.5}}\right](.5) = -.39794,$
$y_3 = -.39794 + \left[\frac{(-.39794)^2}{\sqrt{2}}\right](.5) = -.34195,$
$y_4 = -.34195 + \left[\frac{(-.34195)^2}{\sqrt{2.5}}\right](.5) = -.30497,$
$y_5 = -.27812,\, y_6 = -.25745,\, y_7 = -.24088,\, y_8 = -.2272;$
$\frac{dy}{y^2} = \frac{dx}{\sqrt{x}} \;\Rightarrow\; -\frac{1}{y} = 2\sqrt{x} + C;\, y(1) = -1 \;\Rightarrow\; 1 = 2 + C \;\Rightarrow\; C = -1 \;\Rightarrow\; y = \frac{1}{1 - 2\sqrt{x}} \;\Rightarrow\; y(5) = \frac{1}{1 - 2\sqrt{5}} \approx -.2880$

20. $y_1 = 1 + (0 \cdot \sin 1)\left(\frac{1}{3}\right) = 1,$
$y_2 = 1 + \left(\frac{1}{3} \cdot \sin 1\right)\left(\frac{1}{3}\right) = 1.09350,$
$y_3 = 1.09350 + \left(\frac{2}{3} \cdot \sin 1.09350\right)\left(\frac{1}{3}\right) = 1.29089,$
$y_4 = 1.29089 + \left(\frac{3}{3} \cdot \sin 1.29089\right)\left(\frac{1}{3}\right) = 1.61125,$
$y_5 = 1.61125 + \left(\frac{4}{3} \cdot \sin 1.61125\right)\left(\frac{1}{3}\right) = 2.05533,$
$y_6 = 2.05533 + \left(\frac{5}{3} \cdot \sin 2.05533\right)\left(\frac{1}{3}\right) = 2.54694;$
$y' = x \sin y \Rightarrow \csc y\,dy = x\,dx \Rightarrow -\ln|\csc y + \cot y| = \frac{1}{2}x^2 + C \Rightarrow \csc y + \cot y = e^{-\frac{1}{2}x^2 + C} = Ce^{-\frac{1}{2}x^2}$
$\Rightarrow \frac{1 + \cos y}{\sin y} = Ce^{-\frac{1}{2}x^2} \Rightarrow \cot\left(\frac{y}{2}\right) = Ce^{-\frac{1}{2}x^2};\, y(0) = 1 \Rightarrow \cot\left(\frac{1}{2}\right) = Ce^0 = C \Rightarrow \cot\left(\frac{y}{2}\right) = \cot\left(\frac{1}{2}\right)e^{-\frac{1}{2}x^2}$
$\Rightarrow y = 2\cot^{-1}\left(\cot\left(\frac{1}{2}\right)e^{-\frac{1}{2}x^2}\right),\, y(2) = 2\cot^{-1}\left(\cot\left(\frac{1}{2}\right)e^{-2}\right) = 2.65591$

21. $y = -1 - x + (1 + x_0 + y_0)e^{x - x_0} \Rightarrow y(x_0) = -1 - x_0 + (1 + x_0 + y_0)e^{x_0 - x_0} = -1 - x_0 + (1 + x_0 + y_0)(1) = y_0$
$\frac{dy}{dx} = -1 + (1 + x_0 + y_0)e^{x - x_0} \Rightarrow y = -1 - x + (1 + x_0 + y_0)e^{x - x_0} = \frac{dy}{dx} - x \Rightarrow \frac{dy}{dx} = x + y$

22. $y' = f(x),\ y(x_0) = y_0 \Rightarrow y = \int_{x_0}^{x} f(t)dt + C,\ y(x_0) = \int_{x_0}^{x_0} f(t)dt + C = C \Rightarrow C = y_0 \Rightarrow y = \int_{x_0}^{x} f(t)dt + y_0$

23-34. Example CAS commands:

Maple:

```
ode := diff( y(x), x ) = y(x);
icA := [0, 1];
icB := [0, 2];
icC := [0,-1];
DEplot( ode, y(x), x=0..2, [icA,icB,icC], arrows=slim, linecolor=blue, title="#23 (Section 9.1)" );
```

Mathematica:

To plot vector fields, you must begin by loading a graphics package.

```
<<Graphics`PlotField`
```

To control lengths and appearance of vectors, select the Help browser, type PlotVectorField and select Go.

```
Clear[x, y, f]
yprime = y (2 − y);
pv = PlotVectorField[{1, yprime}, {x, −5, 5}, {y, −4, 6}, Axes → True, AxesLabel → {x, y}];
```

To draw solution curves with Mathematica, you must first solve the differential equation. This will be done with the DSolve command. The y[x] and x at the end of the command specify the dependent and independent variables. The command will not work unless the y in the differential equation is referenced as y[x].

```
equation = y'[x] == y[x] (2 − y[x]) ;
initcond = y[a] == b;
sols = DSolve[{equation, initcond}, y[x], x]
vals = {{0, 1/2}, {0, 3/2},  {0, 2}, {0, 3}}
f[{a_, b_}] = sols[[1, 1, 2]];
solnset = Map[f, vals]
ps = Plot[Evaluate[solnset, {x, −5, 5}];
Show[pv, ps, PlotRange  →  {−4, 6}];
```

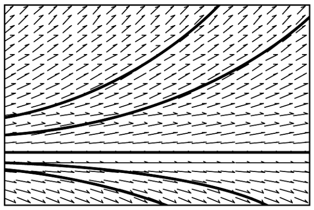
The code for problems such as 31 & 32 is similar for the direction field, but the analytical solutions involve complicated inverse functions, so the numerical solver NDSolve is used. Note that a domain interval is specified.

```
equation = y'[x] == Cos[2x − y[x]] ;
initcond = y[0] == 2;
sol = NDSolve[{equation, initcond}, y[x], {x, 0, 5}]
ps = Plot[Evaluate[y[x]/.sol, {x, 0, 5}];
N[y[x] /. sol/.x → 2]
Show[pv, ps, PlotRange  →  {0, 5}];
```

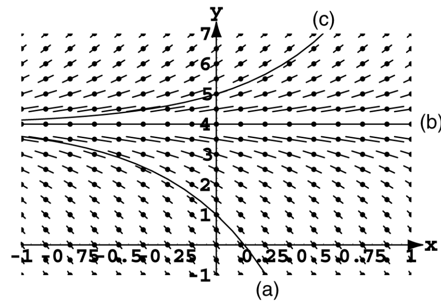
Solutions for 34 can be found one at a time and plots named and shown together. No direction fields here. For 34, the direction field code is similar, but the solution is found implicitly using integrations. The plot requires loading another special graphics package.

```
<<Graphics`ImplicitPlot`
Clear[x,y]
solution[c_] = Integrate[2 (y − 1), y] == Integrate[3x^2 + 4x + 2, x] + c
values =  {−6, −4, −2, 0, 2, 4, 6};
solns = Map[solution, values];
ps = ImplicitPlot[solns, {x, −3, 3}, {y, −3, 3}]
Show[pv, ps]
```

23.



24.



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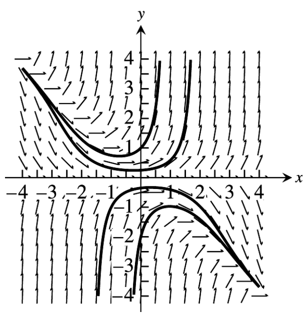

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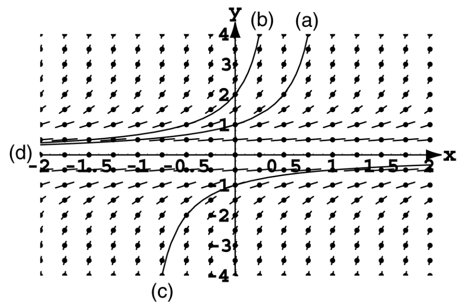
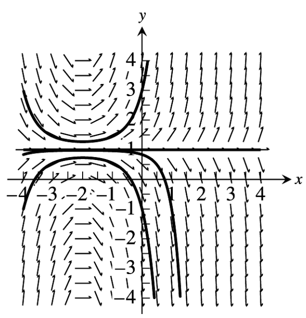
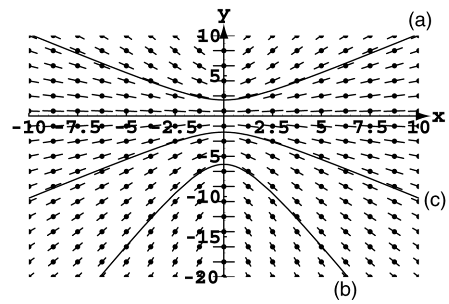


27.



28.



35. $\frac{dy}{dx} = 2xe^{x^2}, y(0) = 2 \Rightarrow y_{n+1} = y_n + 2x_n e^{x_n^2} dx = y_n + 2x_n e^{x_n^2}(0.1) = y_n + 0.2x_n e^{x_n^2}$

On a TI-84 calculator home screen, type the following commands:

2 STO > y: 0 STO > x: y (enter)

y + 0.2*x*e^(x^2) STO > y: x + 0.1 STO > x: y (enter, 10 times)

The last value displayed gives $y_{Euler}(1) \approx 3.45835$

The exact solution: $dy = 2xe^{x^2}dx \Rightarrow y = e^{x^2} + C; y(0) = 2 = e^0 + C \Rightarrow C = 1 \Rightarrow y = 1 + e^{x^2}$

$\Rightarrow y_{exact}(1) = 1 + e \approx 3.71828$

36. $\frac{dy}{dx} = 2y^2(x - 1), y(2) = -\frac{1}{2} \Rightarrow y_{n+1} = y_n + 2y_n^2(x_n - 1)dx = y_n + 0.2\,y_n^2(x_n - 1)$

On a TI-84 calculator home screen, type the following commands:

-0.5 STO > y: 2 STO > x: y (enter)

y + 0.2*y^2(x − 1) STO > y: x + 0.1 STO > x: y (enter, 10 times)

The last value displayed gives $y_{Euler}(2) \approx -0.19285$

The exact solution: $\frac{dy}{dx} = 2y^2(x - 1) \Rightarrow \frac{dy}{y^2} = (2x - 2)dx \Rightarrow -\frac{1}{y} = x^2 - 2x + C \Rightarrow \frac{1}{y} = -x^2 + 2x + C$

$y(2) = -\frac{1}{2} \Rightarrow \frac{1}{-1/2} = -(2)^2 + 2(2) + C = C \Rightarrow C = -2 \Rightarrow \frac{1}{y} = -x^2 + 2x - 2 \Rightarrow y = \frac{1}{-x^2 + 2x - 2}$

$y(3) = \frac{1}{-(3)^2 + 2(3) - 2} = -0.2$

37. $\frac{dy}{dx} = \frac{\sqrt{x}}{y}, y > 0, y(0) = 1 \Rightarrow y_{n+1} = y_n + \frac{\sqrt{x_n}}{y_n}dx = y_n + \frac{\sqrt{x_n}}{y_n}(0.1) = y_n + 0.1\frac{\sqrt{x_n}}{y_n}$
On a TI-84 calculator home screen, type the following commands:
1 STO > y: 0 STO > x: y (enter)
$y + 0.1$*($\sqrt{x}$ /y) STO > y: x + 0.1 STO > x: y (enter, 10 times)
The last value displayed gives $y_{\text{Euler}}(1) \approx 1.5000$
The exact solution: $dy = \frac{\sqrt{x}}{y}dx \Rightarrow y\,dy = \sqrt{x}\,dx \Rightarrow \frac{y^2}{2} = \frac{2}{3}x^{3/2} + C; \frac{(y(0))^2}{2} = \frac{1^2}{2} = \frac{1}{2} = \frac{2}{3}(0)^{3/2} + C \Rightarrow C = \frac{1}{2}$
$\Rightarrow \frac{y^2}{2} = \frac{2}{3}x^{3/2} + \frac{1}{2} \Rightarrow y = \sqrt{\frac{4}{3}x^{3/2} + 1} \Rightarrow y_{\text{exact}}(1) = \sqrt{\frac{4}{3}(1)^{3/2} + 1} \approx 1.5275$

38. $\frac{dy}{dx} = 1 + y^2, y(0) = 0 \Rightarrow y_{n+1} = y_n + (1 + y_n^2)dx = y_n + (1 + y_n^2)(0.1) = y_n + 0.1(1 + y_n^2)$
On a TI-84 calculator home screen, type the following commands:
0 STO > y: 0 STO > x: y (enter)
$y + 0.1$*($1 + y^2$) STO > y: x + 0.1 STO > x: y (enter, 10 times)
The last value displayed gives $y_{\text{Euler}}(1) \approx 1.3964$
The exact solution: $dy = (1 + y^2)dx \Rightarrow \frac{dy}{1+y^2} = dx \Rightarrow \tan^{-1}y = x + C; \tan^{-1}y(0) = \tan^{-1}0 = 0 = 0 + C \Rightarrow C = 0$
$\Rightarrow \tan^{-1}y = x \Rightarrow y = \tan x \Rightarrow y_{\text{exact}}(1) = \tan 1 \approx 1.5574$

39. Example CAS commands:
Maple:

```
ode := diff( y(x), x ) = x + y(x);ic := y(0)=-7/10;
x0 := -4;x1 := 4;y0 := -4; y1 := 4;
b := 1;
P1 := DEplot( ode, y(x), x=x0..x1, y=y0..y1, arrows=thin, title="#39(a) (Section 9.1)" ):
P1;
Ygen := unapply( rhs(dsolve( ode, y(x) )), x,_C1 );                                  # (b)
P2 := seq( plot( Ygen(x,c), x=x0..x1, y=y0..y1, color=blue ), c=-2..2 ):             # (c)
display( [P1,P2], title="#39(c) (Section 9.1)" );
CC := solve( Ygen(0,C)=rhs(ic), C );                                                 # (d)
Ypart := Ygen(x,CC);
P3 := plot( Ypart, x=0..b, title="#39(d) (Section 9.1)" ):
P3;
euler4 := dsolve( {ode,ic}, numeric, method=classical[foreuler], stepsize=(x1-x0)/4 ):   # (e)
P4 := odeplot( euler4, [x,y(x)], x=0..b, numpoints=4, color=blue ):
display( [P3,P4], title="#39(e) (Section 9.1)" );
euler8 := dsolve( {ode,ic}, numeric, method=classical[foreuler], stepsize=(x1-x0)/8 ):   # (f)
P5 := odeplot( euler8, [x,y(x)], x=0..b, numpoints=8, color=green ):
euler16 := dsolve( {ode,ic}, numeric, method=classical[foreuler], stepsize=(x1-x0)/16 ):
P6 := odeplot( euler16, [x,y(x)], x=0..b, numpoints=16, color=pink ):
euler32 := dsolve( {ode,ic}, numeric, method=classical[foreuler], stepsize=(x1-x0)/32 ):
P7 := odeplot( euler32, [x,y(x)], x=0..b, numpoints=32, color=cyan ):
display( [P3,P4,P5,P6,P7], title="#39(f) (Section 9.1)" );
<< N | h        | `percent error` >,                                                 # (g)
 <  4 | (x1-x0)/ 4 | evalf[5]( abs(1-eval(y(x),euler4(b))/eval(Ypart,x=b))*100 ) >,
 <  8 | (x1-x0)/ 8 | evalf[5]( abs(1-eval(y(x),euler8(b))/eval(Ypart,x=b))*100 ) >,
 < 16 | (x1-x0)/16 | evalf[5]( abs(1-eval(y(x),euler16(b))/eval(Ypart,x=b))*100 ) >,
 < 32 | (x1-x0)/32 | evalf[5]( abs(1-eval(y(x),euler32(b))/eval(Ypart,x=b))*100 ) > >;
```

39-42. Example CAS commands:

Mathematica: (assigned functions, step sizes, and values for initial conditions may vary)

Problems 39 - 42 involve use of code from Problems 23 - 34 together with the above code for Euler's method.

9.2 FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

1. $x\,\frac{dy}{dx} + y = e^x \Rightarrow \frac{dy}{dx} + \left(\frac{1}{x}\right) y = \frac{e^x}{x}$, $P(x) = \frac{1}{x}$, $Q(x) = \frac{e^x}{x}$
$\int P(x)\,dx = \int \frac{1}{x}\,dx = \ln|x| = \ln x,\ x > 0 \Rightarrow v(x) = e^{\int P(x)\,dx} = e^{\ln x} = x$
$y = \frac{1}{v(x)} \int v(x)\,Q(x)\,dx = \frac{1}{x}\int x\left(\frac{e^x}{x}\right) dx = \frac{1}{x}\left(e^x + C\right) = \frac{e^x + C}{x},\ x > 0$

2. $e^x\,\frac{dy}{dx} + 2e^x y = 1 \Rightarrow \frac{dy}{dx} + 2y = e^{-x}$, $P(x) = 2$, $Q(x) = e^{-x}$
$\int P(x)\,dx = \int 2\,dx = 2x \Rightarrow v(x) = e^{\int P(x)\,dx} = e^{2x}$
$y = \frac{1}{e^{2x}}\int e^{2x}\cdot e^{-x}\,dx = \frac{1}{e^{2x}}\int e^x\,dx = \frac{1}{e^{2x}}\left(e^x + C\right) = e^{-x} + Ce^{-2x}$

3. $xy' + 3y = \frac{\sin x}{x^2},\ x > 0 \Rightarrow \frac{dy}{dx} + \left(\frac{3}{x}\right) y = \frac{\sin x}{x^3}$, $P(x) = \frac{3}{x}$, $Q(x) = \frac{\sin x}{x^3}$
$\int \frac{3}{x}\,dx = 3\ln|x| = \ln x^3,\ x > 0 \Rightarrow v(x) = e^{\ln x^3} = x^3$
$y = \frac{1}{x^3}\int x^3\left(\frac{\sin x}{x^3}\right) dx = \frac{1}{x^3}\int \sin x\,dx = \frac{1}{x^3}\left(-\cos x + C\right) = \frac{C - \cos x}{x^3},\ x > 0$

4. $y' + (\tan x)\,y = \cos^2 x,\ -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \frac{dy}{dx} + (\tan x)\,y = \cos^2 x$, $P(x) = \tan x$, $Q(x) = \cos^2 x$
$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| = \ln(\cos x)^{-1},\ -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow v(x) = e^{\ln(\cos x)^{-1}} = (\cos x)^{-1}$
$y = \frac{1}{(\cos x)^{-1}}\int (\cos x)^{-1}\cdot\cos^2 x\,dx = (\cos x)\int \cos x\,dx = (\cos x)(\sin x + C) = \sin x\cos x + C\cos x$

5. $x\,\frac{dy}{dx} + 2y = 1 - \frac{1}{x},\ x > 0 \Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right) y = \frac{1}{x} - \frac{1}{x^2}$, $P(x) = \frac{2}{x}$, $Q(x) = \frac{1}{x} - \frac{1}{x^2}$
$\int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2,\ x > 0 \Rightarrow v(x) = e^{\ln x^2} = x^2$
$y = \frac{1}{x^2}\int x^2\left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \frac{1}{x^2}\int (x - 1)\,dx = \frac{1}{x^2}\left(\frac{x^2}{2} - x + C\right) = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2},\ x > 0$

6. $(1 + x)\,y' + y = \sqrt{x} \Rightarrow \frac{dy}{dx} + \left(\frac{1}{1+x}\right) y = \frac{\sqrt{x}}{1+x}$, $P(x) = \frac{1}{1+x}$, $Q(x) = \frac{\sqrt{x}}{1+x}$
$\int \frac{1}{1+x}\,dx = \ln(1 + x)$, since $x > 0 \Rightarrow v(x) = e^{\ln(1+x)} = 1$
$y = \frac{1}{1+x}\int (1 + x)\left(\frac{\sqrt{x}}{1+x}\right) dx = \frac{1}{1+x}\int \sqrt{x}\,dx = \left(\frac{1}{1+x}\right)\left(\frac{2}{3}x^{3/2} + C\right) = \frac{2x^{3/2}}{3(1+x)} + \frac{C}{1+x}$

7. $\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2}e^{x/2} \Rightarrow P(x) = -\frac{1}{2}$, $Q(x) = \frac{1}{2}e^{x/2} \Rightarrow \int P(x)\,dx = -\frac{1}{2}x \Rightarrow v(x) = e^{-x/2}$
$\Rightarrow y = \frac{1}{e^{-x/2}}\int e^{-x/2}\left(\frac{1}{2}e^{x/2}\right) dx = e^{x/2}\int \frac{1}{2}\,dx = e^{x/2}\left(\frac{1}{2}x + C\right) = \frac{1}{2}xe^{x/2} + Ce^{x/2}$

8. $\frac{dy}{dx} + 2y = 2xe^{-2x} \Rightarrow P(x) = 2$, $Q(x) = 2xe^{-2x} \Rightarrow \int P(x)\,dx = \int 2\,dx = 2x \Rightarrow v(x) = e^{2x}$
$\Rightarrow y = \frac{1}{e^{2x}}\int e^{2x}\left(2xe^{-2x}\right) dx = \frac{1}{e^{2x}}\int 2x\,dx = e^{-2x}\left(x^2 + C\right) = x^2e^{-2x} + Ce^{-2x}$

9. $\frac{dy}{dx} - \left(\frac{1}{x}\right) y = 2\ln x \Rightarrow P(x) = -\frac{1}{x}$, $Q(x) = 2\ln x \Rightarrow \int P(x)\,dx = -\int \frac{1}{x}\,dx = -\ln x,\ x > 0$
$\Rightarrow v(x) = e^{-\ln x} = \frac{1}{x} \Rightarrow y = x\int \left(\frac{1}{x}\right)(2\ln x)\,dx = x\left[(\ln x)^2 + C\right] = x(\ln x)^2 + Cx$

10. $\frac{dy}{dx} + \left(\frac{2}{x}\right) y = \frac{\cos x}{x^2}, x > 0 \Rightarrow P(x) = \frac{2}{x}, Q(x) = \frac{\cos x}{x^2} \Rightarrow \int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2, x > 0$
$\Rightarrow v(x) = e^{\ln x^2} = x^2 \Rightarrow y = \frac{1}{x^2}\int x^2\left(\frac{\cos x}{x^2}\right)dx = \frac{1}{x^2}\int \cos x\,dx = \frac{1}{x^2}(\sin x + C) = \frac{\sin x + C}{x^2}$

11. $\frac{ds}{dt} + \left(\frac{4}{t-1}\right) s = \frac{t+1}{(t-1)^3} \Rightarrow P(t) = \frac{4}{t-1}, Q(t) = \frac{t+1}{(t-1)^3} \Rightarrow \int P(t)\,dt = \int \frac{4}{t-1}\,dt = 4\ln|t-1| = \ln(t-1)^4$
$\Rightarrow v(t) = e^{\ln(t-1)^4} = (t-1)^4 \Rightarrow s = \frac{1}{(t-1)^4}\int (t-1)^4\left[\frac{t+1}{(t-1)^3}\right]dt = \frac{1}{(t-1)^4}\int (t^2-1)\,dt$
$= \frac{1}{(t-1)^4}\left(\frac{t^3}{3} - t + C\right) = \frac{t^3}{3(t-1)^4} - \frac{t}{(t-1)^4} + \frac{C}{(t-1)^4}$

12. $(t+1)\frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2} \Rightarrow \frac{ds}{dt} + \left(\frac{2}{t+1}\right)s = 3 + \frac{1}{(t+1)^3} \Rightarrow P(t) = \frac{2}{t+1}, Q(t) = 3 + (t+1)^{-3}$
$\Rightarrow \int P(t)\,dt = \int \frac{2}{t+1}\,dt = 2\ln|t+1| = \ln(t+1)^2 \Rightarrow v(t) = e^{\ln(t+1)^2} = (t+1)^2$
$\Rightarrow s = \frac{1}{(t+1)^2}\int (t+1)^2\left[3 + (t+1)^{-3}\right]dt = \frac{1}{(t+1)^2}\int \left[3(t+1)^2 + (t+1)^{-1}\right]dt$
$= \frac{1}{(t+1)^2}\left[(t+1)^3 + \ln|t+1| + C\right] = (t+1) + (t+1)^{-2}\ln(t+1) + \frac{C}{(t+1)^2}, t > -1$

13. $\frac{dr}{d\theta} + (\cot\theta)r = \sec\theta \Rightarrow P(\theta) = \cot\theta, Q(\theta) = \sec\theta \Rightarrow \int P(\theta)\,d\theta = \int \cot\theta\,d\theta = \ln|\sin\theta| \Rightarrow v(\theta) = e^{\ln|\sin\theta|}$
$= \sin\theta$ because $0 < \theta < \frac{\pi}{2} \Rightarrow r = \frac{1}{\sin\theta}\int (\sin\theta)(\sec\theta)\,d\theta = \frac{1}{\sin\theta}\int \tan\theta\,d\theta = \frac{1}{\sin\theta}(\ln|\sec\theta| + C)$
$= (\csc\theta)(\ln|\sec\theta| + C)$

14. $\tan\theta\frac{dr}{d\theta} + r = \sin^2\theta \Rightarrow \frac{dr}{d\theta} + \frac{r}{\tan\theta} = \frac{\sin^2\theta}{\tan\theta} \Rightarrow \frac{dr}{d\theta} + (\cot\theta)r = \sin\theta\cos\theta \Rightarrow P(\theta) = \cot\theta, Q(\theta) = \sin\theta\cos\theta$
$\Rightarrow \int P(\theta)\,d\theta = \int \cot\theta\,d\theta = \ln|\sin\theta| = \ln(\sin\theta)$ since $0 < \theta < \frac{\pi}{2} \Rightarrow v(\theta) = e^{\ln(\sin\theta)} = \sin\theta$
$\Rightarrow r = \frac{1}{\sin\theta}\int (\sin\theta)(\sin\theta\cos\theta)\,d\theta = \frac{1}{\sin\theta}\int \sin^2\theta\cos\theta\,d\theta = \left(\frac{1}{\sin\theta}\right)\left(\frac{\sin^3\theta}{3} + C\right) = \frac{\sin^2\theta}{3} + \frac{C}{\sin\theta}$

15. $\frac{dy}{dt} + 2y = 3 \Rightarrow P(t) = 2, Q(t) = 3 \Rightarrow \int P(t)\,dt = \int 2\,dt = 2t \Rightarrow v(t) = e^{2t} \Rightarrow y = \frac{1}{e^{2t}}\int 3e^{2t}\,dt$
$= \frac{1}{e^{2t}}\left(\frac{3}{2}e^{2t} + C\right); y(0) = 1 \Rightarrow \frac{3}{2} + C = 1 \Rightarrow C = -\frac{1}{2} \Rightarrow y = \frac{3}{2} - \frac{1}{2}e^{-2t}$

16. $\frac{dy}{dt} + \frac{2y}{t} = t^2 \Rightarrow P(t) = \frac{2}{t}, Q(t) = t^2 \Rightarrow \int P(t)\,dt = 2\ln|t| \Rightarrow v(t) = e^{\ln t^2} = t^2 \Rightarrow y = \frac{1}{t^2}\int (t^2)(t^2)\,dt$
$= \frac{1}{t^2}\int t^4\,dt = \frac{1}{t^2}\left(\frac{t^5}{5} + C\right) = \frac{t^3}{5} + \frac{C}{t^2}; y(2) = 1 \Rightarrow \frac{8}{5} + \frac{C}{4} = 1 \Rightarrow C = -\frac{12}{5} \Rightarrow y = \frac{t^3}{5} - \frac{12}{5t^2}$

17. $\frac{dy}{d\theta} + \left(\frac{1}{\theta}\right)y = \frac{\sin\theta}{\theta} \Rightarrow P(\theta) = \frac{1}{\theta}, Q(\theta) = \frac{\sin\theta}{\theta} \Rightarrow \int P(\theta)\,d\theta = \ln|\theta| \Rightarrow v(\theta) = e^{\ln|\theta|} = |\theta|$
$\Rightarrow y = \frac{1}{|\theta|}\int |\theta|\left(\frac{\sin\theta}{\theta}\right)d\theta = \frac{1}{\theta}\int \theta\left(\frac{\sin\theta}{\theta}\right)d\theta$ for $\theta \neq 0 \Rightarrow y = \frac{1}{\theta}\int \sin\theta\,d\theta = \frac{1}{\theta}(-\cos\theta + C)$
$= -\frac{1}{\theta}\cos\theta + \frac{C}{\theta}; y\left(\frac{\pi}{2}\right) = 1 \Rightarrow C = \frac{\pi}{2} \Rightarrow y = -\frac{1}{\theta}\cos\theta + \frac{\pi}{2\theta}$

18. $\frac{dy}{d\theta} - \left(\frac{2}{\theta}\right)y = \theta^2\sec\theta\tan\theta \Rightarrow P(\theta) = -\frac{2}{\theta}, Q(\theta) = \theta^2\sec\theta\tan\theta \Rightarrow \int P(\theta)\,d\theta = -2\ln|\theta| \Rightarrow v(\theta) = e^{-2\ln|\theta|}$
$= \theta^{-2} \Rightarrow y = \frac{1}{\theta^{-2}}\int (\theta^{-2})(\theta^2\sec\theta\tan\theta)\,d\theta = \theta^2\int \sec\theta\tan\theta\,d\theta = \theta^2(\sec\theta + C) = \theta^2\sec\theta + C\theta^2;$
$y\left(\frac{\pi}{3}\right) = 2 \Rightarrow 2 = \left(\frac{\pi^2}{9}\right)(2) + C\left(\frac{\pi^2}{9}\right) \Rightarrow C = \frac{18}{\pi^2} - 2 \Rightarrow y = \theta^2\sec\theta + \left(\frac{18}{\pi^2} - 2\right)\theta^2$

19. $(x+1)\frac{dy}{dx} - 2(x^2 + x)y = \frac{e^{x^2}}{x+1} \Rightarrow \frac{dy}{dx} - 2\left[\frac{x(x+1)}{x+1}\right]y = \frac{e^{x^2}}{(x+1)^2} \Rightarrow \frac{dy}{dx} - 2xy = \frac{e^{x^2}}{(x+1)^2} \Rightarrow P(x) = -2x,$
$Q(x) = \frac{e^{x^2}}{(x+1)^2} \Rightarrow \int P(x)\,dx = \int -2x\,dx = -x^2 \Rightarrow v(x) = e^{-x^2} \Rightarrow y = \frac{1}{e^{-x^2}}\int e^{-x^2}\left[\frac{e^{x^2}}{(x+1)^2}\right]dx$
$= e^{x^2}\int \frac{1}{(x+1)^2}\,dx = e^{x^2}\left[\frac{(x+1)^{-1}}{-1} + C\right] = -\frac{e^{x^2}}{x+1} + Ce^{x^2}; y(0) = 5 \Rightarrow -\frac{1}{0+1} + C = 5 \Rightarrow -1 + C = 5$
$\Rightarrow C = 6 \Rightarrow y = 6e^{x^2} - \frac{e^{x^2}}{x+1}$

20. $\frac{dy}{dx} + xy = x \Rightarrow P(x) = x, Q(x) = x \Rightarrow \int P(x)\,dx = \int x\,dx = \frac{x^2}{2} \Rightarrow v(x) = e^{x^2/2} \Rightarrow y = \frac{1}{e^{x^2/2}} \int e^{x^2/2} \cdot x\,dx$
$= \frac{1}{e^{x^2/2}} \left(e^{x^2/2} + C\right) = 1 + \frac{C}{e^{x^2/2}}$; $y(0) = -6 \Rightarrow 1 + C = -6 \Rightarrow C = -7 \Rightarrow y = 1 - \frac{7}{e^{x^2/2}}$

21. $\frac{dy}{dt} - ky = 0 \Rightarrow P(t) = -k, Q(t) = 0 \Rightarrow \int P(t)\,dt = \int -k\,dt = -kt \Rightarrow v(t) = e^{-kt}$
$\Rightarrow y = \frac{1}{e^{-kt}} \int \left(e^{-kt}\right)(0)\,dt = e^{kt}(0 + C) = Ce^{kt}$; $y(0) = y_0 \Rightarrow C = y_0 \Rightarrow y = y_0 e^{kt}$

22. (a) $\frac{du}{dt} + \frac{k}{m} u = 0 \Rightarrow P(t) = \frac{k}{m}, Q(t) = 0 \Rightarrow \int P(t)\,dt = \int \frac{k}{m}\,dt = \frac{k}{m} t = \frac{kt}{m} \Rightarrow u(t) = e^{kt/m}$
$\Rightarrow y = \frac{1}{e^{kt/m}} \int e^{kt/m} \cdot 0\,dt = \frac{C}{e^{kt/m}}$; $u(0) = u_0 \Rightarrow \frac{C}{e^{k(0)/m}} = u_0 \Rightarrow C = u_0 \Rightarrow u = u_0\, e^{-(k/m)t}$
(b) $\frac{du}{dt} = -\frac{k}{m} u \Rightarrow \frac{du}{u} = -\frac{k}{m}\,dt \Rightarrow \ln u = -\frac{k}{m} t + C \Rightarrow u = e^{-(k/m)t + C} \Rightarrow u = e^{-(k/m)t} \cdot e^C$. Let $e^C = C_1$.
Then $u = \frac{1}{e^{(k/m)t}} \cdot C_1$ and $u(0) = u_0 = \frac{1}{e^{(k/m)(0)}} \cdot C_1 = C_1$. So $u = u_0\, e^{-(k/m)t}$

23. $x \int \frac{1}{x}\,dx = x(\ln|x| + C) = x \ln|x| + Cx \Rightarrow$ (b) is correct

24. $\frac{1}{\cos x} \int \cos x\,dx = \frac{1}{\cos x}(\sin x + C) = \tan x + \frac{C}{\cos x} \Rightarrow$ (b) is correct

25. Steady State $= \frac{V}{R}$ and we want $i = \frac{1}{2}\left(\frac{V}{R}\right) \Rightarrow \frac{1}{2}\left(\frac{V}{R}\right) = \frac{V}{R}\left(1 - e^{-Rt/L}\right) \Rightarrow \frac{1}{2} = 1 - e^{-Rt/L} \Rightarrow -\frac{1}{2} = -e^{-Rt/L}$
$\Rightarrow \ln \frac{1}{2} = -\frac{Rt}{L} \Rightarrow -\frac{L}{R} \ln \frac{1}{2} = t \Rightarrow t = \frac{L}{R} \ln 2$ sec

26. (a) $\frac{di}{dt} + \frac{R}{L} i = 0 \Rightarrow \frac{1}{i}\,di = -\frac{R}{L}\,dt \Rightarrow \ln i = -\frac{Rt}{L} + C_1 \Rightarrow i = e^{C_1} e^{-Rt/L} = Ce^{-Rt/L}$; $i(0) = I \Rightarrow I = C$
$\Rightarrow i = Ie^{-Rt/L}$ amp
(b) $\frac{1}{2} I = I\, e^{-Rt/L} \Rightarrow e^{-Rt/L} = \frac{1}{2} \Rightarrow -\frac{Rt}{L} = \ln \frac{1}{2} = -\ln 2 \Rightarrow t = \frac{L}{R} \ln 2$ sec
(c) $t = \frac{L}{R} \Rightarrow i = I\, e^{(-Rt/L)(L/R)} = I\, e^{-t}$ amp

27. (a) $t = \frac{3L}{R} \Rightarrow i = \frac{V}{R}\left(1 - e^{(-R/L)(3L/R)}\right) = \frac{V}{R}\left(1 - e^{-3}\right) \approx 0.9502\, \frac{V}{R}$ amp, or about 95% of the steady state value
(b) $t = \frac{2L}{R} \Rightarrow i = \frac{V}{R}\left(1 - e^{(-R/L)(2L/R)}\right) = \frac{V}{R}\left(1 - e^{-2}\right) \approx 0.8647\, \frac{V}{R}$ amp, or about 86% of the steady state value

28. (a) $\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L} \Rightarrow P(t) = \frac{R}{L}, Q(t) = \frac{V}{L} \Rightarrow \int P(t)\,dt = \int \frac{R}{L}\,dt = \frac{Rt}{L} \Rightarrow v(t) = e^{Rt/L}$
$\Rightarrow i = \frac{1}{e^{Rt/L}} \int e^{Rt/L}\left(\frac{V}{L}\right) dt = \frac{1}{e^{Rt/L}}\left[\frac{L}{R} e^{Rt/L}\left(\frac{V}{L}\right) + C\right] = \frac{V}{R} + Ce^{-(R/L)t}$
(b) $i(0) = 0 \Rightarrow \frac{V}{R} + C = 0 \Rightarrow C = -\frac{V}{R} \Rightarrow i = \frac{V}{R} - \frac{V}{R} e^{-Rt/L}$
(c) $i = \frac{V}{R} \Rightarrow \frac{di}{dt} = 0 \Rightarrow \frac{di}{dt} + \frac{R}{L} i = 0 + \left(\frac{R}{L}\right)\left(\frac{V}{R}\right) = \frac{V}{L} \Rightarrow i = \frac{V}{R}$ is a solution of Eq. (11); $i = Ce^{-(R/L)t}$

29. $y' - y = -y^2$; we have $n = 2$, so let $u = y^{1-2} = y^{-1}$. Then $y = u^{-1}$ and $\frac{du}{dx} = -1y^{-2} \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -y^2 \frac{du}{dx}$
$\Rightarrow -u^{-2} \frac{du}{dx} - u^{-1} = -u^{-2} \Rightarrow \frac{du}{dx} + u = 1$. With $e^{\int dx} = e^x$ as the integrating factor, we have
$e^x\left(\frac{du}{dx} + u\right) = \frac{d}{dx}(e^x u) = e^x$. Integrating, we get $e^x u = e^x + C \Rightarrow u = 1 + \frac{C}{e^x} = \frac{1}{y} \Rightarrow y = \frac{1}{1 + \frac{C}{e^x}} = \frac{e^x}{e^x + C}$

30. $y' - y = xy^2$; we have $n = 2$, so let $u = y^{-1}$. Then $y = u^{-1}$ and $\frac{du}{dx} = -y^{-2} \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -y^2 \frac{du}{dx} = -u^{-2} \frac{du}{dx}$.
Substituting: $-u^{-2} \frac{du}{dx} - u^{-1} = xu^{-2} \Rightarrow \frac{du}{dx} + u = -x$. Using $e^{\int dx} = e^x$ as an integrating factor:
$e^x\left(\frac{du}{dx} + u\right) = \frac{d}{dx}(e^x u) = -x\, e^x \Rightarrow e^x u = e^x(1 - x) + C \Rightarrow u = \frac{e^x(1 - x) + C}{e^x} \Rightarrow y = u^{-1} = \frac{e^x}{e^x - xe^x + C}$

31. $xy' + y = y^{-2} \Rightarrow y' + \left(\frac{1}{x}\right)y = \left(\frac{1}{x}\right)y^{-2}$. Let $u = y^{1-(-2)} = y^3 \Rightarrow y = u^{1/3}$ and $y^{-2} = u^{-2/3}$.
$\frac{du}{dx} = 3y^2 \frac{dy}{dx} \Rightarrow y' = \frac{dy}{dx} = \left(\frac{1}{3}\right)\left(\frac{du}{dx}\right)\left(y^{-2}\right) = \left(\frac{1}{3}\right)\left(\frac{du}{dx}\right)\left(u^{-2/3}\right)$. Thus we have

$\left(\frac{1}{3}\right)\left(\frac{du}{dx}\right)\left(u^{-2/3}\right) + \left(\frac{1}{x}\right)u^{1/3} = \left(\frac{1}{x}\right)u^{-2/3} \Rightarrow \frac{du}{dx} + \left(\frac{3}{x}\right)u = \left(\frac{3}{x}\right)1$. The integrating factor, $v(x)$, is $e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$. Thus $\frac{d}{dx}(x^3u) = \left(\frac{3}{x}\right)x^3 = 3x^2 \Rightarrow x^3u = x^3 + C \Rightarrow u = 1 + \frac{C}{x^3} = y^3$ $\Rightarrow y = \left(1 + \frac{C}{x^3}\right)^{1/3}$

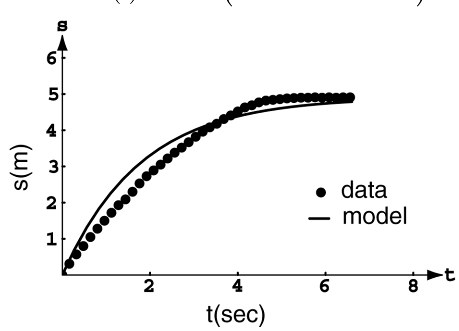
32. $x^2\, y' + 2xy = y^3 \Rightarrow y' + \left(\frac{2}{x}\right)y = \left(\frac{1}{x^2}\right)y^3$. $P(x) = \left(\frac{2}{x}\right)$, $Q(x) = \left(\frac{1}{x^2}\right)$, $n = 3$. Let $u = y^{1-3} = y^{-2}$. Substituting gives $\frac{du}{dx} + (-2)\left(\frac{2}{x}\right)u = -2\left(\frac{1}{x^2}\right) \Rightarrow \frac{du}{dx} + \left(\frac{-4}{x}\right)u = \frac{-2}{x^2}$. Let the integrating factor, $v(x)$, be $e^{\int\left(\frac{-4}{x}\right)dx} = e^{\ln x^{-4}} = x^{-4}$. Thus $\frac{d}{dx}(x^{-4}\,u) = -2x^{-6} \Rightarrow x^{-4}\,u = \frac{2}{5}x^{-5} + C \Rightarrow u = \frac{2}{5x}+Cx^4 = y^{-2}$ $\Rightarrow y = \left(\frac{2}{5x}+Cx^4\right)^{-1/2}$

9.3 APPLICATIONS

1. Note that the total mass is $66 + 7 = 73$ kg, therefore, $v = v_0 e^{-(k/m)t} \Rightarrow v = 9e^{-3.9t/73}$
   (a) $s(t) = \int 9e^{-3.9t/73}dt = -\frac{2190}{13}e^{-3.9t/73} + C$
   Since $s(0) = 0$ we have $C = \frac{2190}{13}$ and $\lim\limits_{t\to\infty} s(t) = \lim\limits_{t\to\infty} \frac{2190}{13}\left(1 - e^{-3.9t/73}\right) = \frac{2190}{13} \approx 168.5$
   The cyclist will coast about 168.5 meters.
   (b) $1 = 9e^{-3.9t/73} \Rightarrow \frac{3.9t}{73} = \ln 9 \Rightarrow t = \frac{73\ln 9}{3.9} \approx 41.13$ sec
   It will take about 41.13 seconds.

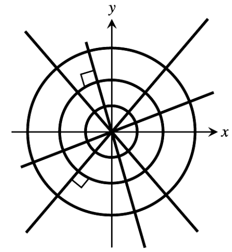
2. $v = v_0 e^{-(k/m)t} \Rightarrow v = 9e^{-(59{,}000/51{,}000{,}000)t} \Rightarrow v = 9e^{-59t/51{,}000}$
   (a) $s(t) = \int 9e^{-59t/51{,}000}dt = -\frac{459{,}0000}{59}\, e^{-59t/51{,}000} + C$
   Since $s(0) = 0$ we have $C = \frac{459{,}0000}{59}$ and $\lim\limits_{t\to\infty} s(t) = \lim\limits_{t\to\infty} \frac{459{,}0000}{59}\left(1 - e^{-59t/51{,}000}\right) = \frac{459{,}0000}{59} \approx 7780$ m
   The ship will coast about 7780 m, or 7.78 km.
   (b) $1 = 9e^{-59t/51{,}000} \Rightarrow \frac{59t}{51{,}000} = \ln 9 \Rightarrow t = \frac{51{,}000\ln 9}{59} \approx 1899.3$ sec
   It will take about 31.65 minutes.

3. The total distance traveled $= \frac{v_0 m}{k} \Rightarrow \frac{(2.75)(39.92)}{k} = 4.91 \Rightarrow k = 22.36$. Therefore, the distance traveled is given by the function $s(t) = 4.91\left(1 - e^{-(22.36/39.92)t}\right)$. The graph shows $s(t)$ and the data points.

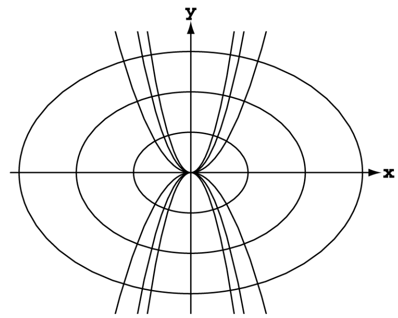


4. $\frac{v_0 m}{k}$ = coasting distance $\Rightarrow \frac{(0.80)(49.90)}{k} = 1.32 \Rightarrow k = \frac{998}{33}$
   We know that $\frac{v_0 m}{k} = 1.32$ and $\frac{k}{m} = \frac{998}{33(49.9)} = \frac{20}{33}$.
   Using Equation 3, we have: $s(t) = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right) = 1.32\left(1 - e^{-20t/33}\right) \approx 1.32(1 - e^{-0.606t})$

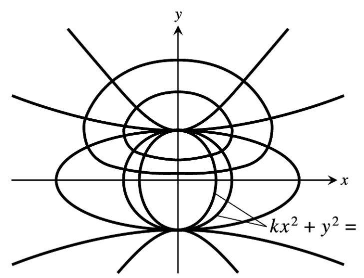
5. $y = mx \Rightarrow \frac{y}{x} = m \Rightarrow \frac{xy' - y}{x^2} = 0 \Rightarrow y' = \frac{y}{x}$. So for orthogonals: $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y\,dy = -x\,dx \Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = C$ $\Rightarrow x^2 + y^2 = C_1$



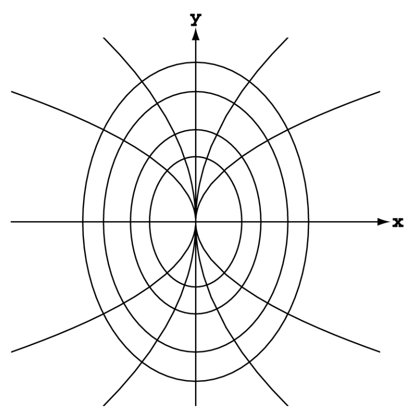
6. $y = cx^2 \Rightarrow \frac{y}{x^2} = c \Rightarrow \frac{x^2y' - 2xy}{x^4} = 0 \Rightarrow x^2y' = 2xy$ $\Rightarrow y' = \frac{2y}{x}$. So for the orthogonals: $\frac{dy}{dx} = -\frac{x}{2y}$ $\Rightarrow 2ydy = -xdx \Rightarrow y^2 = -\frac{x^2}{2} + C \Rightarrow y = \pm\sqrt{\frac{x^2}{2} + C}$, $C > 0$



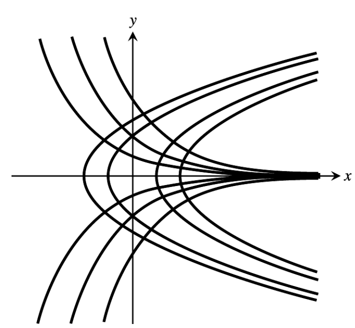
7. $kx^2 + y^2 = 1 \Rightarrow 1 - y^2 = kx^2 \Rightarrow \frac{1-y^2}{x^2} = k$ $\Rightarrow \frac{x^2(2y)y' - (1-y^2)2x}{x^4} = 0 \Rightarrow -2yx^2y' = (1-y^2)(2x)$ $\Rightarrow y' = \frac{(1-y^2)(2x)}{-2xy^2} = \frac{(1-y^2)}{-xy}$. So for the orthogonals: $\frac{dy}{dx} = \frac{xy}{1-y^2} \Rightarrow \frac{(1-y^2)}{y}dy = x\,dx \Rightarrow \ln y - \frac{y^2}{2} = \frac{x^2}{2} + C$



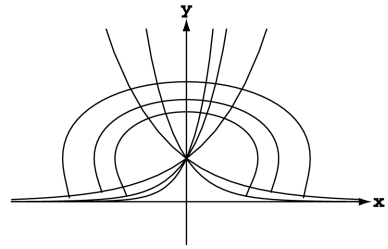
8. $2x^2 + y^2 = c^2 \Rightarrow 4x + 2yy' = 0 \Rightarrow y' = -\frac{4x}{2y} = -\frac{2x}{y}$. For orthogonals: $\frac{dy}{dx} = \frac{y}{2x} \Rightarrow \frac{dy}{y} = \frac{dx}{2x} \Rightarrow \ln y = \frac{1}{2}\ln x + C$ $\Rightarrow \ln y = \ln x^{1/2} + \ln C_1 \Rightarrow y = C_1\,|x|^{1/2}$



9. $y = ce^{-x} \Rightarrow \frac{y}{e^{-x}} = c \Rightarrow \frac{e^{-x}y' - y(e^{-x})(-1)}{(e^{-x})^2} = 0$ $\Rightarrow e^{-x}y' = -ye^{-x} \Rightarrow y' = -y$. So for the orthogonals: $\frac{dy}{dx} = \frac{1}{y} \Rightarrow y\,dy = dx \Rightarrow \frac{y^2}{2} = x + C$ $\Rightarrow y^2 = 2x + C_1 \Rightarrow y = \pm\sqrt{2x + C_1}$

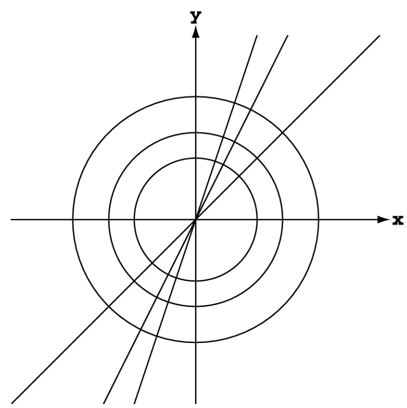


10. $y = e^{kx} \Rightarrow \ln y = kx \Rightarrow \frac{\ln y}{x} = k \Rightarrow \frac{x\left(\frac{1}{y}\right)y' - \ln y}{x^2} = 0$
$\Rightarrow \left(\frac{x}{y}\right) y' - \ln y = 0 \Rightarrow y' = \frac{y \ln y}{x}$. So for the orthogonals:
$\frac{dy}{dx} = \frac{-x}{y \ln y} \Rightarrow y \ln y\, dy = -x\, dx$
$\Rightarrow \frac{1}{2}y^2 \ln y - \frac{1}{4}(y^2) = \left(-\frac{1}{2}x^2\right) + C$
$\Rightarrow y^2 \ln y - \frac{y^2}{2} = -x^2 + C_1$

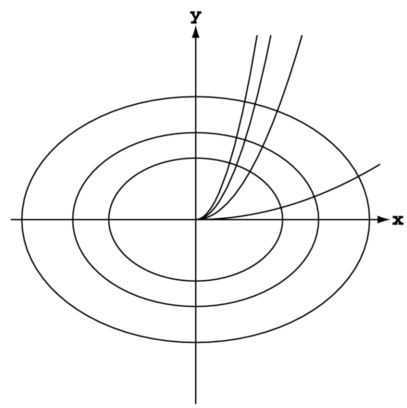


11. $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ intersect at $(1, 1)$. Also, $2x^2 + 3y^2 = 5 \Rightarrow 4x + 6y\, y' = 0 \Rightarrow y' = -\frac{4x}{6y} \Rightarrow y'(1, 1) = -\frac{2}{3}$
$y_1^2 = x^3 \Rightarrow 2y_1 y_1' = 3x^2 \Rightarrow y_1' = \frac{3x^2}{2y_1} \Rightarrow y_1'(1, 1) = \frac{3}{2}$. Since $y' \cdot y_1' = \left(-\frac{2}{3}\right)\left(\frac{3}{2}\right) = -1$, the curves are orthogonal.

12. (a) $x\, dx + y\, dy = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = C$ is the general equation of the family with slope $y' = -\frac{x}{y}$. For the orthogonals: $y' = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln y = \ln x + C$ or $y = C_1 x$ (where $C_1 = e^C$) is the general equation of the orthogonals.



(b) $x\, dy - 2y\, dx = 0 \Rightarrow 2y\, dx = x\, dy \Rightarrow \frac{dy}{2y} = \frac{dx}{x}$
$\Rightarrow \frac{1}{2}\left(\frac{dy}{y}\right) = \frac{dx}{x} \Rightarrow \frac{1}{2}\ln y = \ln x + C \Rightarrow y = C_1 x^2$ is the equation for the solution family.
$\frac{1}{2}\ln y - \ln x = C \Rightarrow \frac{1}{2}\frac{y'}{y} - \frac{1}{x} = 0 \Rightarrow y' = \frac{2y}{x}$
$\Rightarrow$ slope of orthogonals is $\frac{dy}{dx} = -\frac{x}{2y}$
$\Rightarrow 2y\, dy = -x\, dx \Rightarrow y^2 = -\frac{x^2}{2} + C$ is the general equation of the orthogonals.



13. Let $y(t)$ = the amount of salt in the container and $V(t)$ = the total volume of liquid in the tank at time t. Then, the departure rate is $\frac{y(t)}{V(t)}$ (the outflow rate).

(a) Rate entering $= \frac{2\text{ lb}}{\text{gal}} \cdot \frac{5\text{ gal}}{\text{min}} = 10$ lb/min

(b) Volume $= V(t) = 100$ gal $+ (5t$ gal $- 4t$ gal$) = (100 + t)$ gal

(c) The volume at time t is $(100 + t)$ gal. The amount of salt in the tank at time t is y lbs. So the concentration at any time t is $\frac{y}{100+t}$ lbs/gal. Then, the rate leaving $= \frac{y}{100+t}$ (lbs/gal) $\cdot$ 4 (gal/min) $= \frac{4y}{100+t}$ lbs/min

(d) $\frac{dy}{dt} = 10 - \frac{4y}{100+t} \Rightarrow \frac{dy}{dt} + \left(\frac{4}{100+t}\right) y = 10 \Rightarrow P(t) = \frac{4}{100+t}$, $Q(t) = 10 \Rightarrow \int P(t)\, dt = \int \frac{4}{100+t}\, dt$
$= 4 \ln(100 + t) \Rightarrow v(t) = e^{4 \ln(100+t)} = (100 + t)^4 \Rightarrow y = \frac{1}{(100+t)^4} \int (100 + t)^4 (10\, dt)$
$= \frac{10}{(100+t)^4}\left(\frac{(100+t)^5}{5} + C\right) = 2(100 + t) + \frac{C}{(100+t)^4}$; $y(0) = 50 \Rightarrow 2(100 + 0) + \frac{C}{(100+0)^4} = 50$
$\Rightarrow C = -(150)(100)^4 \Rightarrow y = 2(100 + t) - \frac{(150)(100)^4}{(100+t)^4} \Rightarrow y = 2(100 + t) - \frac{150}{\left(1 + \frac{t}{100}\right)^4}$

(e) $y(25) = 2(100+25) - \frac{(150)(100)^4}{(100+25)^4} \approx 188.56$ lbs $\Rightarrow$ concentration $= \frac{y(25)}{\text{volume}} \approx \frac{188.6}{125} \approx 1.5$ lb/gal

14. (a) $\frac{dV}{dt} = (5-3) = 2 \Rightarrow V = 100 + 2t$

The tank is full when $V = 200 = 100 + 2t \Rightarrow t = 50$ min

(b) Let $y(t)$ be the amount of concentrate in the tank at time t.

$\frac{dy}{dt} = \left(\frac{1}{2}\frac{\text{lb}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) - \left(\frac{y}{100+2t}\frac{\text{lb}}{\text{gal}}\right)\left(3\frac{\text{gal}}{\text{min}}\right) \Rightarrow \frac{dy}{dt} = \frac{5}{2} - \frac{3}{2}\left(\frac{y}{50+t}\right) \Rightarrow \frac{dy}{dt} + \frac{3}{2(t+50)}y = \frac{5}{2}$

$Q(t) = \frac{5}{2}$; $P(t) = \frac{3}{2}\left(\frac{1}{50+t}\right) \Rightarrow \int P(t)\,dt = \frac{3}{2}\int \frac{1}{t+50}\,dt = \frac{3}{2}\ln(t+50)$ since $t + 50 > 0$

$v(t) = e^{\int P(t)\,dt} = e^{\frac{3}{2}\ln(t+50)} = (t+50)^{3/2}$

$y(t) = \frac{1}{(t+50)^{3/2}}\int \frac{5}{2}(t+50)^{3/2}\,dt = (t+50)^{-3/2}\left[(t+50)^{5/2} + C\right] \Rightarrow y(t) = t + 50 + \frac{C}{(t+50)^{3/2}}$

Apply the initial condition (i.e., distilled water in the tank at $t = 0$):

$y(0) = 0 = 50 + \frac{C}{50^{3/2}} \Rightarrow C = -50^{5/2} \Rightarrow y(t) = t + 50 - \frac{50^{5/2}}{(t+50)^{3/2}}$. When the tank is full at $t = 50$,

$y(50) = 100 - \frac{50^{5/2}}{100^{3/2}} \approx 83.22$ pounds of concentrate.

15. Let y be the amount of fertilizer in the tank at time t. Then rate entering $= 1\,\frac{\text{lb}}{\text{gal}} \cdot 1\,\frac{\text{gal}}{\text{min}} = 1\,\frac{\text{lb}}{\text{min}}$ and the volume in the tank at time t is $V(t) = 100$ (gal) + [1 (gal/min) − 3 (gal/min)]t min = (100 − 2t) gal. Hence rate out $= \left(\frac{y}{100-2t}\right)3 = \frac{3y}{100-2t}$ lbs/min $\Rightarrow \frac{dy}{dt} = \left(1 - \frac{3y}{100-2t}\right)$ lbs/min $\Rightarrow \frac{dy}{dt} + \left(\frac{3}{100-2t}\right)y = 1$

$\Rightarrow P(t) = \frac{3}{100-2t}$, $Q(t) = 1 \Rightarrow \int P(t)\,dt = \int \frac{3}{100-2t}\,dt = \frac{3\ln(100-2t)}{-2} \Rightarrow v(t) = e^{(-3\ln(100-2t))2}$

$= (100-2t)^{-3/2} \Rightarrow y = \frac{1}{(100-2t)^{-3/2}}\int (100-2t)^{-3/2}\,dt = (100-2t)^{-3/2}\left[\frac{-2(100-2t)^{-1/2}}{-2} + C\right]$

$= (100-2t) + C(100-2t)^{3/2}$; $y(0) = 0 \Rightarrow [100 - 2(0)] + C[100 - 2(0)]^{3/2} \Rightarrow C(100)^{3/2} = -100$

$\Rightarrow C = -(100)^{-1/2} = -\frac{1}{10} \Rightarrow y = (100-2t) - \frac{(100-2t)^{3/2}}{10}$. Let $\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -2 - \frac{\left(\frac{3}{2}\right)(100-2t)^{1/2}(-2)}{10}$

$= -2 + \frac{3\sqrt{100-2t}}{10} = 0 \Rightarrow 20 = 3\sqrt{100-2t} \Rightarrow 400 = 9(100-2t) \Rightarrow 400 = 900 - 18t \Rightarrow -500 = -18t$

$\Rightarrow t \approx 27.8$ min, the time to reach the maximum. The maximum amount is then

$y(27.8) = [100 - 2(27.8)] - \frac{[100-2(27.8)]^{3/2}}{10} \approx 14.8$ lb

16. Let $y = y(t)$ be the amount of carbon monoxide (CO) in the room at time t. The amount of CO entering the room is $\left(\frac{4}{100} \times \frac{3}{10}\right) = \frac{12}{1000}$ ft^3/min, and the amount of CO leaving the room is $\left(\frac{y}{4500}\right)\left(\frac{3}{10}\right) = \frac{y}{15{,}000}$ ft^3/min. Thus, $\frac{dy}{dt} = \frac{12}{1000} - \frac{y}{15{,}000} \Rightarrow \frac{dy}{dt} + \frac{1}{15{,}000}y = \frac{12}{1000} \Rightarrow P(t) = \frac{1}{15{,}000}$, $Q(t) = \frac{12}{1000} \Rightarrow v(t) = e^{t/15{,}000}$

$\Rightarrow y = \frac{1}{e^{t/15{,}000}}\int \frac{12}{1000}e^{t/15{,}000}\,dt \Rightarrow y = e^{-t/15{,}000}\left(\frac{12 \cdot 15{,}000}{1000}e^{t/15{,}000} + C\right) = e^{-t/15{,}000}\left(180e^{t/15{,}000} + C\right)$;

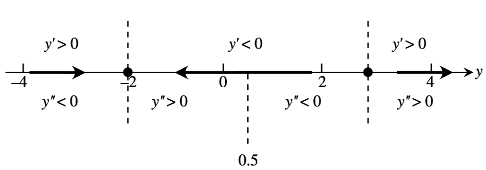
$y(0) = 0 \Rightarrow 0 = 1(180 + C) \Rightarrow C = -180 \Rightarrow y = 180 - 180e^{-t/15{,}000}$. When the concentration of CO is 0.01% in the room, the amount of CO satisfies $\frac{y}{4500} = \frac{.01}{100} \Rightarrow y = 0.45$ ft^3. When the room contains this amount we have $0.45 = 180 - 180e^{-t/15{,}000} \Rightarrow \frac{179.55}{180} = e^{-t/15{,}000} \Rightarrow t = -15{,}000\ln\left(\frac{179.55}{180}\right) \approx 37.55$ min.

9.4 GRAPHICAL SOUTIONS OF AUTONOMOUS EQUATIONS

1. $y' = (y+2)(y-3)$

(a) $y = -2$ is a stable equilibrium value and $y = 3$ is an unstable equilibrium.

(b) $y'' = (2y-1)y' = 2(y+2)\left(y - \frac{1}{2}\right)(y-3)$



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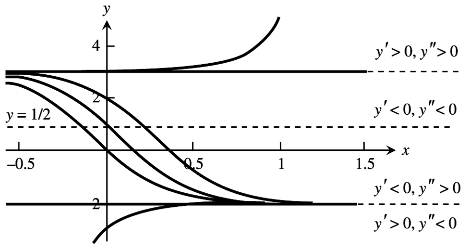
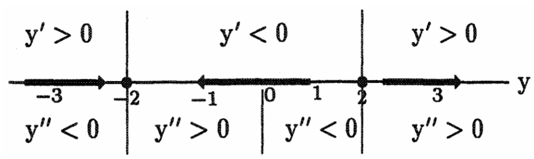


2. $y' = (y + 2)(y - 2)$
   (a) $y = -2$ is a stable equilibrium value and $y = 2$ is an unstable equilibrium.
   (b) $y'' = 2yy' = 2(y + 2)y(y - 2)$



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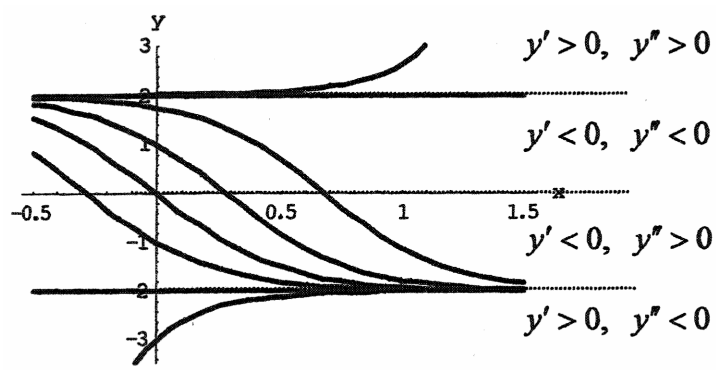
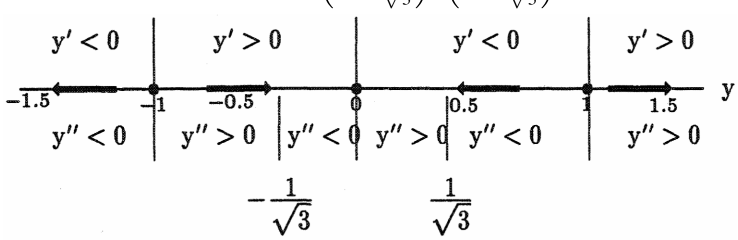


3. $y' = y^3 - y = (y + 1)y(y - 1)$
   (a) $y = -1$ and $y = 1$ is an unstable equilibrium and $y = 0$ is a stable equilibrium value.
   (b) $y'' = (3y^2 - 1)y' = 3(y + 1)\left(y + \frac{1}{\sqrt{3}}\right)y\left(y - \frac{1}{\sqrt{3}}\right)(y - 1)$



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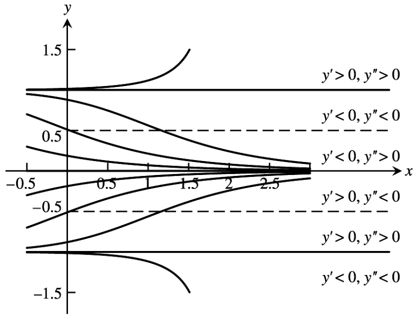
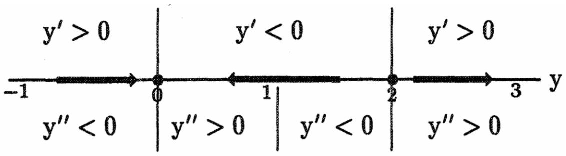


4. $y' = y(y-2)$
   (a) $y = 0$ is a stable equilibrium value and $y = 2$ is an unstable equilibrium.
   (b) $y'' = (2y-2)y' = 2y(y-1)(y-2)$



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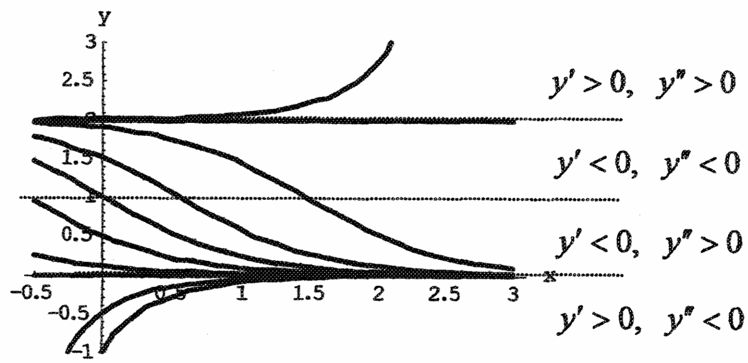
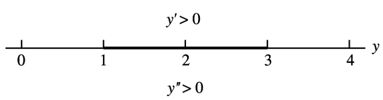


5. $y' = \sqrt{y}, y > 0$
   (a) There are no equilibrium values.
   (b) $y'' = \frac{1}{2\sqrt{y}}\, y' = \frac{1}{2\sqrt{y}}\sqrt{y} = \frac{1}{2}$



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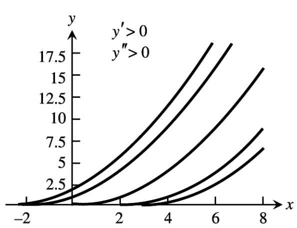
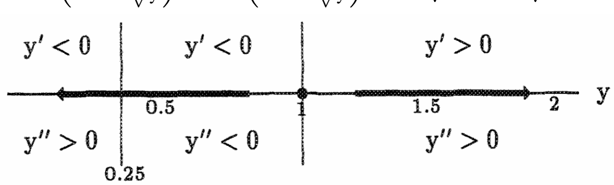


6. $y' = y - \sqrt{y}, y > 0$
   (a) $y = 1$ is an unstable equilibrium.
   (b) $y'' = \left(1 - \frac{1}{2\sqrt{y}}\right) y' = \left(1 - \frac{1}{2\sqrt{y}}\right)\left(y - \sqrt{y}\right) = \left(\sqrt{y} - \frac{1}{2}\right)\left(\sqrt{y} - 1\right)$



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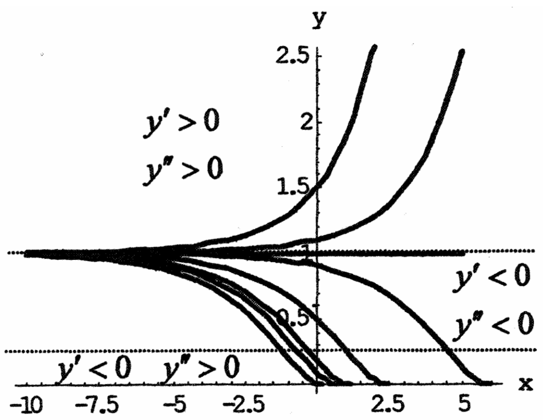
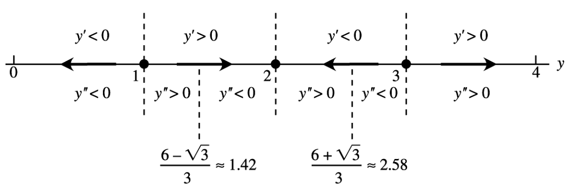


7. $y' = (y-1)(y-2)(y-3)$
   (a) $y = 1$ and $y = 3$ is an unstable equilibrium and $y = 2$ is a stable equilibrium value.
   (b) $y'' = (3y^2 - 12y + 11)(y-1)(y-2)(y-3) = 3(y-1)\left(y - \frac{6-\sqrt{3}}{3}\right)(y-2)\left(y - \frac{6+\sqrt{3}}{3}\right)(y-3)$



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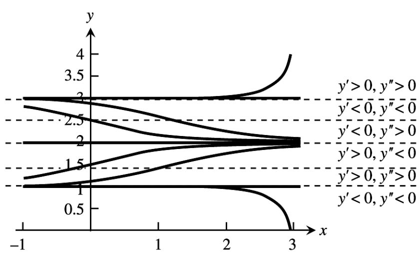
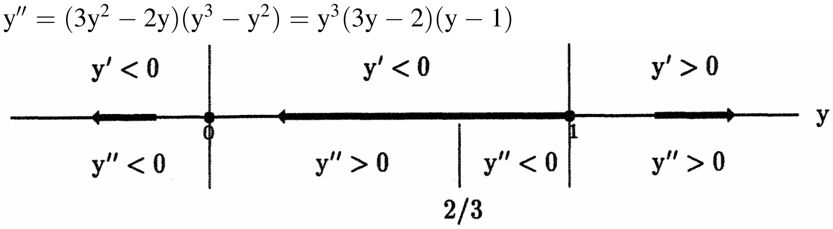


8. $y' = y^3 - y^2 = y^2(y-1)$
   (a) $y = 0$ and $y = 1$ is an unstable equilibrium.
   (b) $y'' = (3y^2 - 2y)(y^3 - y^2) = y^3(3y-2)(y-1)$



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