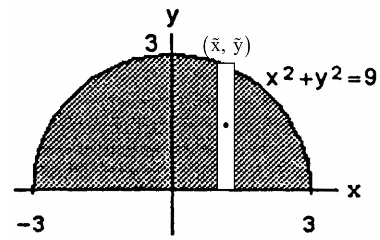
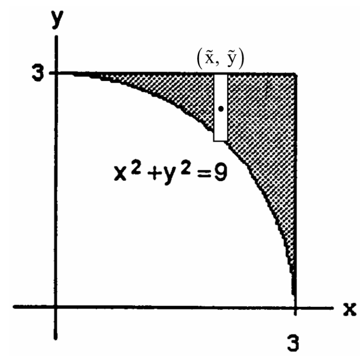
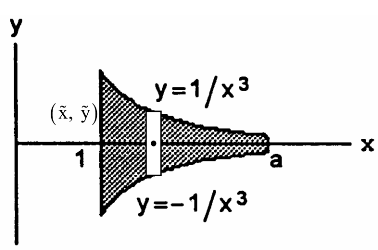
(b) Applying the symmetry argument analogous to the one used in Exercise 1, we find that $\overline{x} = 0$. The typical vertical strip has the same parameters as in part (a). Thus, $M_x = \int \widetilde{y}\ dm = \int_{-3}^{3} \frac{\delta}{2}\left(9 - x^2\right) dx$ $= 2\int_0^3 \frac{\delta}{2}\left(9 - x^2\right) dx = 2(9\delta) = 18\delta$; $M = \int dm = \int \delta\ dA = \delta \int dA$ $= \delta(\text{Area of a semi-circle of radius } 3) = \delta\left(\frac{9\pi}{2}\right) = \frac{9\pi\delta}{2}$. Therefore, $\overline{y} = \frac{M_x}{M} = (18\delta)\left(\frac{2}{9\pi\delta}\right) = \frac{4}{\pi}$, the same $\overline{y}$ as in part (a) $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(0, \frac{4}{\pi}\right)$ is the center of mass.



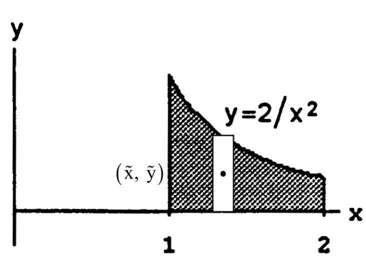
11. Since the plate is symmetric about the line $x = y$ and its density is constant, the distribution of mass is symmetric about this line. This means that $\overline{x} = \overline{y}$. The typical *vertical* strip has
center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{3+\sqrt{9-x^2}}{2}\right)$,
length: $3 - \sqrt{9 - x^2}$, width: dx,
area: $dA = \left(3 - \sqrt{9 - x^2}\right) dx$,
mass: $dm = \delta\ dA = \delta\left(3 - \sqrt{9 - x^2}\right) dx$.
The moment about the x-axis is
$\widetilde{y}\ dm = \delta\,\frac{\left(3+\sqrt{9-x^2}\right)\left(3-\sqrt{9-x^2}\right)}{2}\,dx = \frac{\delta}{2}\left[9 - \left(9 - x^2\right)\right] dx = \frac{\delta x^2}{2}\,dx$. Thus, $M_x = \int_0^3 \frac{\delta x^2}{2}\,dx = \frac{\delta}{6}\left[x^3\right]_0^3 = \frac{9\delta}{2}$. The area equals the area of a square with side length 3 minus one quarter the area of a disk with radius 3 $\Rightarrow$ $A = 3^2 - \frac{\pi 9}{4}$ $= \frac{9}{4}(4 - \pi)$ $\Rightarrow$ $M = \delta A = \frac{9\delta}{4}(4 - \pi)$. Therefore, $\overline{y} = \frac{M_x}{M} = \left(\frac{9\delta}{2}\right)\left[\frac{4}{9\delta(4-\pi)}\right] = \frac{2}{4-\pi}$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{2}{4-\pi}, \frac{2}{4-\pi}\right)$ is the center of mass.



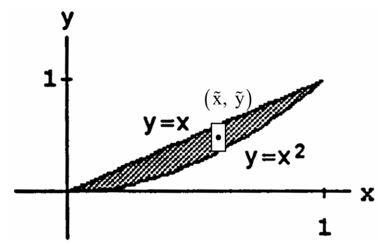
12. Applying the symmetry argument analogous to the one used in Exercise 1, we find that $\overline{y} = 0$. The typical *vertical* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{\frac{1}{x^3} - \frac{1}{x^3}}{2}\right) = (x, 0)$,
length: $\frac{1}{x^3} - \left(-\frac{1}{x^3}\right) = \frac{2}{x^3}$, width: dx, area: $dA = \frac{2}{x^3}\,dx$,
mass: $dm = \delta\ dA = \frac{2\delta}{x^3}\,dx$. The moment about the y-axis is
$\widetilde{x}\ dm = x \cdot \frac{2\delta}{x^3}\,dx = \frac{2\delta}{x^2}\,dx$. Thus, $M_y = \int \widetilde{x}\ dm = \int_1^a \frac{2\delta}{x^2}\,dx$ $= 2\delta\left[-\frac{1}{x}\right]_1^a = 2\delta\left(-\frac{1}{a} + 1\right) = \frac{2\delta(a-1)}{a}$; $M = \int dm = \int_1^a \frac{2\delta}{x^3}\,dx = \delta\left[-\frac{1}{x^2}\right]_1^a = \delta\left(-\frac{1}{a^2} + 1\right) = \frac{\delta\left(a^2-1\right)}{a^2}$. Therefore, $\overline{x} = \frac{M_y}{M} = \left[\frac{2\delta(a-1)}{a}\right]\left[\frac{a^2}{\delta\left(a^2-1\right)}\right] = \frac{2a}{a+1}$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{2a}{a+1}, 0\right)$. Also, $\lim_{a \to \infty} \overline{x} = 2$.



13. $M_x = \int \widetilde{y}\ dm = \int_1^2 \frac{\left(\frac{2}{x^2}\right)}{2} \cdot \delta \cdot \left(\frac{2}{x^2}\right) dx$
$= \int_1^2 \left(\frac{1}{x^2}\right)\left(x^2\right)\left(\frac{2}{x^2}\right) dx = \int_1^2 \frac{2}{x^2}\,dx = 2\int_1^2 x^{-2}\,dx$
$= 2\left[-x^{-1}\right]_1^2 = 2\left[\left(-\frac{1}{2}\right) - (-1)\right] = 2\left(\frac{1}{2}\right) = 1$;
$M_y = \int \widetilde{x}\ dm = \int_1^2 x \cdot \delta \cdot \left(\frac{2}{x^2}\right) dx$
$= \int_1^2 x\left(x^2\right)\left(\frac{2}{x^2}\right) dx = 2\int_1^2 x\,dx = 2\left[\frac{x^2}{2}\right]_1^2$
$= 2\left(2 - \frac{1}{2}\right) = 4 - 1 = 3$; $M = \int dm = \int_1^2 \delta\left(\frac{2}{x^2}\right) dx = \int_1^2 x^2\left(\frac{2}{x^2}\right) dx = 2\int_1^2 dx = 2[x]_1^2 = 2(2-1) = 2$. So $\overline{x} = \frac{M_y}{M} = \frac{3}{2}$ and $\overline{y} = \frac{M_x}{M} = \frac{1}{2}$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{3}{2}, \frac{1}{2}\right)$ is the center of mass.



14. We use the *vertical* strip approach:



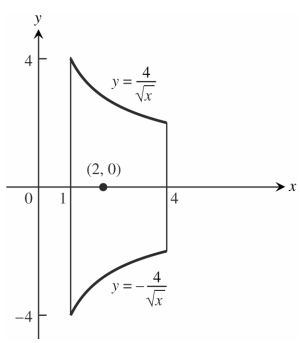
$M_x = \int \tilde{y}\, dm = \int_0^1 \frac{(x+x^2)}{2}(x - x^2) \cdot \delta\, dx$
$= \frac{1}{2}\int_0^1 (x^2 - x^4) \cdot 12x\, dx$
$= 6\int_0^1 (x^3 - x^5)\, dx = 6\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$
$= 6\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{6}{4} - 1 = \frac{1}{2}$;

$M_y = \int \tilde{x}\, dm = \int_0^1 x(x - x^2) \cdot \delta\, dx = \int_0^1 (x^2 - x^3) \cdot 12x\, dx = 12\int_0^1 (x^3 - x^4)\, dx = 12\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 12\left(\frac{1}{4} - \frac{1}{5}\right)$
$= \frac{12}{20} = \frac{3}{5}$; $M = \int dm = \int_0^1 (x - x^2) \cdot \delta\, dx = 12\int_0^1 (x^2 - x^3)\, dx = 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 12\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{12}{12} = 1$. So
$\overline{x} = \frac{M_y}{M} = \frac{3}{5}$ and $\overline{y} = \frac{M_x}{M} = \frac{1}{2} \Rightarrow \left(\frac{3}{5}, \frac{1}{2}\right)$ is the center of mass.

15. (a) We use the shell method: $V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_1^4 2\pi x\left[\frac{4}{\sqrt{x}} - \left(-\frac{4}{\sqrt{x}}\right)\right] dx = 16\pi\int_1^4 \frac{x}{\sqrt{x}}\, dx$
$= 16\pi\int_1^4 x^{1/2}\, dx = 16\pi\left[\frac{2}{3}x^{3/2}\right]_1^4 = 16\pi\left(\frac{2}{3} \cdot 8 - \frac{2}{3}\right) = \frac{32\pi}{3}(8 - 1) = \frac{224\pi}{3}$

(b) Since the plate is symmetric about the x-axis and its density $\delta(x) = \frac{1}{x}$ is a function of x alone, the distribution of its mass is symmetric about the x-axis. This means that $\overline{y} = 0$. We use the vertical strip approach to find $\overline{x}$:
$M_y = \int \tilde{x}\, dm = \int_1^4 x \cdot \left[\frac{4}{\sqrt{x}} - \left(-\frac{4}{\sqrt{x}}\right)\right] \cdot \delta\, dx = \int_1^4 x \cdot \frac{8}{\sqrt{x}} \cdot \frac{1}{x}\, dx = 8\int_1^4 x^{-1/2}\, dx = 8\left[2x^{1/2}\right]_1^4 = 8(2 \cdot 2 - 2) = 16$;
$M = \int dm = \int_1^4 \left[\frac{4}{\sqrt{x}} - \left(\frac{-4}{\sqrt{x}}\right)\right] \cdot \delta\, dx = 8\int_1^4 \left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{x}\right) dx = 8\int_1^4 x^{-3/2}\, dx = 8\left[-2x^{-1/2}\right]_1^4 = 8[-1 - (-2)] = 8$.
So $\overline{x} = \frac{M_y}{M} = \frac{16}{8} = 2 \Rightarrow (\overline{x}, \overline{y}) = (2, 0)$ is the center of mass.

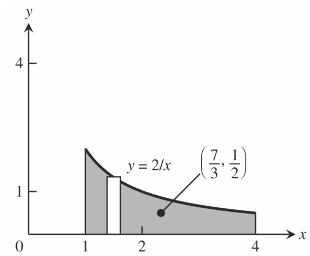
(c)



16. (a) We use the disk method: $V = \int_a^b \pi R^2(x)\, dx = \int_1^4 \pi\left(\frac{4}{x^2}\right) dx = 4\pi\int_1^4 x^{-2}\, dx = 4\pi\left[-\frac{1}{x}\right]_1^4 = 4\pi\left[\frac{-1}{4} - (-1)\right]$
$= \pi[-1 + 4] = 3\pi$

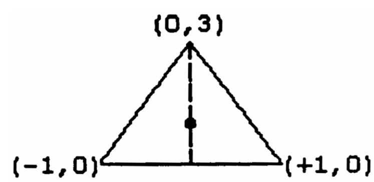
(b) We model the distribution of mass with vertical strips: $M_x = \int \tilde{y}\, dm = \int_1^4 \frac{\left(\frac{2}{x}\right)}{2} \cdot \left(\frac{2}{x}\right) \cdot \delta\, dx = \int_1^4 \frac{2}{x^2} \cdot \sqrt{x}\, dx$
$= 2\int_1^4 x^{-3/2}\, dx = 2\left[\frac{-2}{\sqrt{x}}\right]_1^4 = 2[-1 - (-2)] = 2$; $M_y = \int \tilde{x}\, dm = \int_1^4 x \cdot \frac{2}{x} \cdot \delta\, dx = 2\int_1^4 x^{1/2}\, dx = 2\left[\frac{2x^{3/2}}{3}\right]_1^4 =$
$2\left[\frac{16}{3} - \frac{2}{3}\right] = \frac{28}{3}$; $M = \int dm = \int_1^4 \frac{2}{x} \cdot \delta\, dx = 2\int_1^4 \frac{\sqrt{x}}{x}\, dx = 2\int_1^4 x^{-1/2}\, dx = 2\left[2x^{1/2}\right]_1^4 = 2(4 - 2) = 4$. So
$\overline{x} = \frac{M_y}{M} = \frac{\left(\frac{28}{3}\right)}{4} = \frac{7}{3}$ and $\overline{y} = \frac{M_x}{M} = \frac{2}{4} = \frac{1}{2} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{7}{3}, \frac{1}{2}\right)$ is the center of mass.

(c)

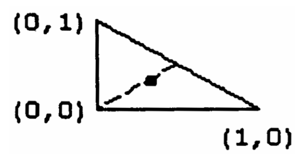


17. The mass of a horizontal strip is $dm = \delta\, dA = \delta L\, dy$, where L is the width of the triangle at a distance of y above its base on the x-axis as shown in the figure in the text. Also, by similar triangles we have $\frac{L}{b} = \frac{h-y}{h}$ $\Rightarrow L = \frac{b}{h}(h-y)$. Thus, $M_x = \int \tilde{y}\, dm = \int_0^h \delta y \left(\frac{b}{h}\right)(h-y)\, dy = \frac{\delta b}{h}\int_0^h (hy - y^2)\, dy = \frac{\delta b}{h}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h$ $= \frac{\delta b}{h}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \delta b h^2 \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\delta b h^2}{6}$; $M = \int dm = \int_0^h \delta \left(\frac{b}{h}\right)(h-y)\, dy = \frac{\delta b}{h}\int_0^h (h-y)\, dy = \frac{\delta b}{h}\left[hy - \frac{y^2}{2}\right]_0^h$ $= \frac{\delta b}{h}\left(h^2 - \frac{h^2}{2}\right) = \frac{\delta b h}{2}$. So $\overline{y} = \frac{M_x}{M} = \left(\frac{\delta b h^2}{6}\right)\left(\frac{2}{\delta b h}\right) = \frac{h}{3}$ $\Rightarrow$ the center of mass lies above the base of the triangle one-third of the way toward the opposite vertex. Similarly the other two sides of the triangle can be placed on the x-axis and the same results will occur. Therefore the centroid does lie at the intersection of the medians, as claimed.

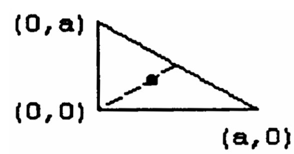
18. From the symmetry about the y-axis it follows that $\overline{x} = 0$. It also follows that the line through the points $(0,0)$ and $(0,3)$ is a median $\Rightarrow$ $\overline{y} = \frac{1}{3}(3-0) = 1$ $\Rightarrow$ $(\overline{x}, \overline{y}) = (0, 1)$.



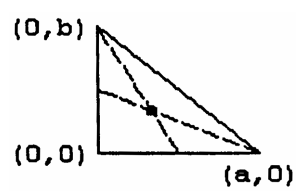
19. From the symmetry about the line $x = y$ it follows that $\overline{x} = \overline{y}$. It also follows that the line through the points $(0,0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a median $\Rightarrow$ $\overline{y} = \overline{x} = \frac{2}{3} \cdot \left(\frac{1}{2} - 0\right) = \frac{1}{3}$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \frac{1}{3}\right)$.



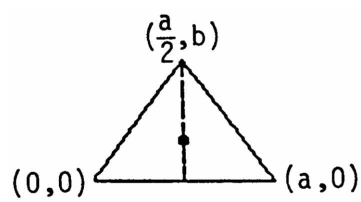
20. From the symmetry about the line $x = y$ it follows that $\overline{x} = \overline{y}$. It also follows that the line through the point $(0,0)$ and $\left(\frac{a}{2}, \frac{a}{2}\right)$ is a median $\Rightarrow$ $\overline{y} = \overline{x} = \frac{2}{3}\left(\frac{a}{2} - 0\right) = \frac{1}{3}a$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{a}{3}, \frac{a}{3}\right)$.



21. The point of intersection of the median from the vertex $(0, b)$ to the opposite side has coordinates $\left(0, \frac{a}{2}\right)$ $\Rightarrow$ $\overline{y} = (b-0) \cdot \frac{1}{3} = \frac{b}{3}$ and $\overline{x} = \left(\frac{a}{2} - 0\right) \cdot \frac{2}{3} = \frac{a}{3}$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{a}{3}, \frac{b}{3}\right)$.



22. From the symmetry about the line $x = \frac{a}{2}$ it follows that $\overline{x} = \frac{a}{2}$. It also follows that the line through the points $\left(\frac{a}{2}, 0\right)$ and $\left(\frac{a}{2}, b\right)$ is a median $\Rightarrow$ $\overline{y} = \frac{1}{3}(b-0) = \frac{b}{3}$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{a}{2}, \frac{b}{3}\right)$.



23. $y = x^{1/2} \Rightarrow dy = \frac{1}{2}x^{-1/2}\,dx$

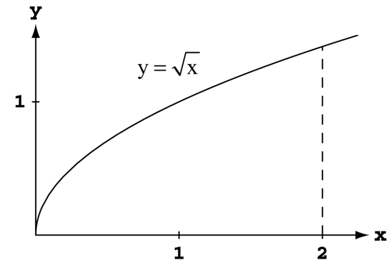
$\Rightarrow ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \frac{1}{4x}}\,dx$;

$M_x = \delta \int_0^2 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx$

$= \delta \int_0^2 \sqrt{x + \frac{1}{4}}\,dx = \frac{2\delta}{3}\left[\left(x + \frac{1}{4}\right)^{3/2}\right]_0^2$

$= \frac{2\delta}{3}\left[\left(2 + \frac{1}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right]$

$= \frac{2\delta}{3}\left[\left(\frac{9}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right] = \frac{2\delta}{3}\left(\frac{27}{8} - \frac{1}{8}\right) = \frac{13\delta}{6}$



24. $y = x^3 \Rightarrow dy = 3x^2\,dx$

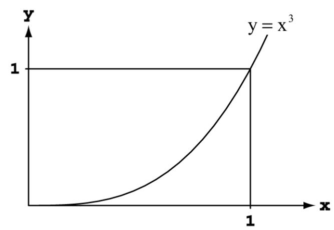
$\Rightarrow dx = \sqrt{(dx)^2 + \left(3x^2\,dx\right)^2} = \sqrt{1 + 9x^4}\,dx$;

$M_x = \delta \int_0^1 x^3\sqrt{1 + 9x^4}\,dx$;

$[u = 1 + 9x^4 \Rightarrow du = 36x^3\,dx \Rightarrow \frac{1}{36}\,du = x^3\,dx$;

$x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 10]$

$\rightarrow M_x = \delta \int_1^{10} \frac{1}{36}u^{1/2}\,du = \frac{\delta}{36}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{\delta}{54}\left(10^{3/2} - 1\right)$



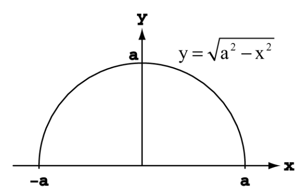
25. From Example 4 we have $M_x = \int_0^\pi a(a\sin\theta)(k\sin\theta)\,d\theta = a^2k\int_0^\pi \sin^2\theta\,d\theta = \frac{a^2k}{2}\int_0^\pi (1 - \cos 2\theta)\,d\theta = \frac{a^2k}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi$
$= \frac{a^2k\pi}{2}$; $M_y = \int_0^\pi a(a\cos\theta)(k\sin\theta)\,d\theta = a^2k\int_0^\pi \sin\theta\cos\theta\,d\theta = \frac{a^2k}{2}\left[\sin^2\theta\right]_0^\pi = 0$; $M = \int_0^\pi ak\sin\theta\,d\theta = ak[-\cos\theta]_0^\pi$
$= 2ak$. Therefore, $\overline{x} = \frac{M_y}{M} = 0$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{a^2k\pi}{2}\right)\left(\frac{1}{2ak}\right) = \frac{a\pi}{4} \Rightarrow \left(0, \frac{a\pi}{4}\right)$ is the center of mass.

26. $M_x = \int \widetilde{y}\,dm = \int_0^\pi (a\sin\theta)\cdot\delta\cdot a\,d\theta$

$= \int_0^\pi (a^2\sin\theta)\,(1 + k\,|\cos\theta|)\,d\theta$

$= a^2\int_0^{\pi/2} (\sin\theta)(1 + k\cos\theta)\,d\theta$

$\quad + a^2\int_{\pi/2}^{\pi} (\sin\theta)(1 - k\cos\theta)\,d\theta$



$= a^2\int_0^{\pi/2} \sin\theta\,d\theta + a^2k\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta + a^2\int_{\pi/2}^{\pi} \sin\theta\,d\theta - a^2k\int_{\pi/2}^{\pi} \sin\theta\cos\theta\,d\theta$

$= a^2[-\cos\theta]_0^{\pi/2} + a^2k\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} + a^2[-\cos\theta]_{\pi/2}^{\pi} - a^2k\left[\frac{\sin^2\theta}{2}\right]_{\pi/2}^{\pi}$

$= a^2[0 - (-1)] + a^2k\left(\frac{1}{2} - 0\right) + a^2[-(-1) - 0] - a^2k\left(0 - \frac{1}{2}\right) = a^2 + \frac{a^2k}{2} + a^2 + \frac{a^2k}{2} = 2a^2 + a^2k = a^2(2 + k)$;

$M_y = \int \widetilde{x}\,dm = \int_0^\pi (a\cos\theta)\cdot\delta\cdot a\,d\theta = \int_0^\pi (a^2\cos\theta)\,(1 + k\,|\cos\theta|)\,d\theta$

$= a^2\int_0^{\pi/2} (\cos\theta)(1 + k\cos\theta)\,d\theta + a^2\int_{\pi/2}^{\pi} (\cos\theta)(1 - k\cos\theta)\,d\theta$

$= a^2\int_0^{\pi/2} \cos\theta\,d\theta + a^2k\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right)d\theta + a^2\int_{\pi/2}^{\pi} \cos\theta\,d\theta - a^2k\int_{\pi/2}^{\pi} \left(\frac{1 + \cos 2\theta}{2}\right)d\theta$

$= a^2[\sin\theta]_0^{\pi/2} + \frac{a^2k}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + a^2[\sin\theta]_{\pi/2}^{\pi} - \frac{a^2k}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{\pi}$

$= a^2(1 - 0) + \frac{a^2k}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 + 0)\right] + a^2(0 - 1) - \frac{a^2k}{2}\left[(\pi + 0) - \left(\frac{\pi}{2} + 0\right)\right] = a^2 + \frac{a^2k\pi}{4} - a^2 - \frac{a^2k\pi}{4} = 0$;

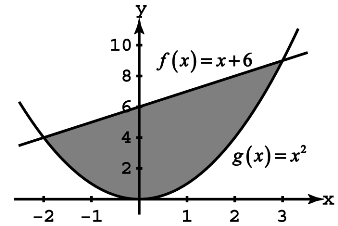
$M = \int_0^\pi \delta\cdot a\,d\theta = a\int_0^\pi (1 + k\,|\cos\theta|)\,d\theta = a\int_0^{\pi/2} (1 + k\cos\theta)\,d\theta + a\int_{\pi/2}^{\pi} (1 - k\cos\theta)\,d\theta$

$= a[\theta + k\sin\theta]_0^{\pi/2} + a[\theta - k\sin\theta]_{\pi/2}^{\pi} = a\left[\left(\frac{\pi}{2} + k\right) - 0\right] + a\left[(\pi + 0) - \left(\frac{\pi}{2} - k\right)\right]$

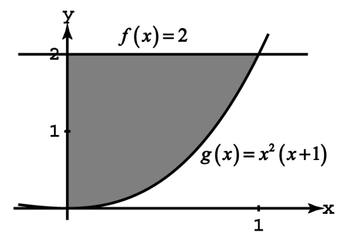
$= \frac{a\pi}{2} + ak + a\left(\frac{\pi}{2} + k\right) = a\pi + 2ak = a(\pi + 2k)$. So $\overline{x} = \frac{M_y}{M} = 0$ and $\overline{y} = \frac{M_x}{M} = \frac{a^2(2 + k)}{a(\pi + 2k)} = \frac{a(2 + k)}{\pi + 2k}$

$\Rightarrow \left(0, \frac{2a + ka}{\pi + 2k}\right)$ is the center of mass.

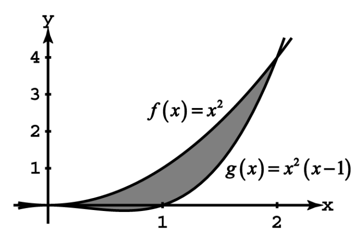
27. $f(x) = x + 6$, $g(x) = x^2$, $f(x) = g(x) \Rightarrow x + 6 = x^2$
$\Rightarrow x^2 - x - 6 = 0 \Rightarrow x = 3, x = -2; \delta = 1$
$M = \int_{-2}^{3} [(x+6) - x^2]dx = \left[\frac{1}{2}x^2 + 6x - \frac{1}{3}x^3\right]_{-2}^{3}$
$= \left(\frac{9}{2} + 18 - 9\right) - \left(2 - 12 + \frac{8}{3}\right) = \frac{125}{6}$
$\overline{x} = \frac{1}{125/6}\int_{-2}^{3} x[(x+6) - x^2]dx = \frac{6}{125}\int_{-2}^{3}[x^2 + 6x - x^3]dx$
$= \frac{6}{125}\left[\frac{1}{3}x^3 + 3x^2 - \frac{1}{4}x^4\right]_{-2}^{3}$
$= \frac{6}{125}\left(9 + 27 - \frac{81}{4}\right) - \frac{6}{125}\left(-\frac{8}{3} + 12 - 4\right) = \frac{1}{2}$; $\overline{y} = \frac{1}{125/6}\int_{-2}^{3}\frac{1}{2}\left[(x+6)^2 - (x^2)^2\right]dx = \frac{3}{125}\int_{-2}^{3}[x^2 + 12x + 36 - x^4]dx$
$= \frac{3}{125}\left[\frac{1}{3}x^3 + 6x^2 + 36x - \frac{1}{5}x^5\right]_{-2}^{3} = \frac{3}{125}\left(9 + 54 + 108 - \frac{243}{5}\right) - \frac{3}{125}\left(-\frac{8}{3} + 24 - 72 + \frac{32}{5}\right) = 4$
$\Rightarrow \left(\frac{1}{2}, 4\right)$ is the center of mass.



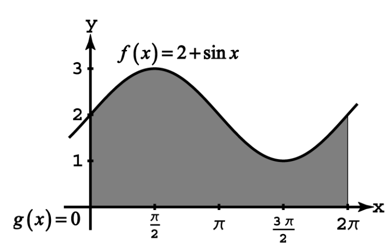
28. $f(x) = 2$, $g(x) = x^2(x+1)$, $f(x) = g(x) \Rightarrow 2 = x^2(x+1)$
$\Rightarrow x^3 + x^2 - 2 = 0 \Rightarrow x = 1; \delta = 1$
$M = \int_0^1 [2 - x^2(x+1)]\,dx = \int_0^1 [2 - x^3 - x^2]\,dx$
$= \left[2x - \frac{1}{4}x^4 - \frac{1}{3}x^3\right]_0^1 = \left(2 - \frac{1}{4} - \frac{1}{3}\right) - 0 = \frac{17}{12}$
$\overline{x} = \frac{1}{17/12}\int_0^1 x[2 - x^2(x+1)]dx = \frac{12}{17}\int_0^1 [2x - x^4 - x^3]dx$
$= \frac{12}{17}\left[x^2 - \frac{1}{5}x^5 - \frac{1}{4}x^4\right]_0^1$
$= \frac{12}{17}\left(1 - \frac{1}{5} - \frac{1}{4}\right) - 0 = \frac{33}{85}$; $\overline{y} = \frac{1}{17/12}\int_0^1 \frac{1}{2}\left[2^2 - \left(x^2(x+1)\right)^2\right]dx = \frac{6}{17}\int_0^1[4 - x^6 - 2x^5 - x^4]dx$
$= \frac{6}{17}\left[4x - \frac{1}{7}x^7 - \frac{1}{3}x^6 - \frac{1}{5}x^5\right]_0^1 = \frac{6}{17}\left(4 - \frac{1}{7} - \frac{1}{3} - \frac{1}{5}\right) - 0 = \frac{698}{595} \Rightarrow \left(\frac{33}{85}, \frac{698}{595}\right)$ is the center of mass.



29. $f(x) = x^2$, $g(x) = x^2(x-1)$, $f(x) = g(x) \Rightarrow x^2 = x^2(x-1)$
$\Rightarrow x^3 - 2x^2 = 0 \Rightarrow x = 0, x = 2; \delta = 1$
$M = \int_0^2 [x^2 - x^2(x-1)]dx = \int_0^2 [2x^2 - x^3]dx$
$= \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \left(\frac{16}{3} - 4\right) - 0 = \frac{4}{3}$
$\overline{x} = \frac{1}{4/3}\int_0^2 x[x^2 - x^2(x-1)]dx = \frac{3}{4}\int_0^2[2x^3 - x^4]dx$
$= \frac{3}{4}\left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{3}{4}\left(8 - \frac{32}{5}\right) - 0 = \frac{6}{5}$;
$\overline{y} = \frac{1}{4/3}\int_0^2 \frac{1}{2}\left[(x^2)^2 - \left(x^2(x-1)\right)^2\right]dx = \frac{3}{8}\int_0^2[2x^5 - x^6]dx = \frac{3}{8}\left[\frac{1}{3}x^6 - \frac{1}{7}x^7\right]_0^2 = \frac{3}{8}\left(\frac{64}{3} - \frac{128}{7}\right) - 0 = \frac{8}{7}$
$\Rightarrow \left(\frac{6}{5}, \frac{8}{7}\right)$ is the center of mass.



30. $f(x) = 2 + \sin x$, $g(x) = 0$, $x = 0$, $x = 2\pi$; $\delta = 1$;
$M = \int_0^{2\pi}[2 + \sin x]dx = [2x - \cos x]_0^{2\pi}$
$= (4\pi - 1) - (0 - 1) = 4\pi$
$\overline{x} = \frac{1}{4\pi}\int_0^{2\pi} x[2 + \sin x - 0]dx = \frac{1}{4\pi}\int_0^{2\pi}[2x + x\sin x]dx$
$= \frac{1}{4\pi}\int_0^{2\pi} 2x\,dx + \frac{1}{4\pi}\int_0^{2\pi} x\sin x dx$
$= \frac{1}{4\pi}[x^2]_0^{2\pi} + \frac{1}{4\pi}[\sin x - x\cos x]_0^{2\pi}$
$= \frac{1}{4\pi}(4\pi^2) - 0 + \frac{1}{4\pi}(0 - 2\pi) - 0 = \frac{2\pi - 1}{2}$; $\overline{y} = \frac{1}{4\pi}\int_0^{2\pi}\frac{1}{2}\left[(2 + \sin x)^2 - (0)^2\right]dx = \frac{1}{8\pi}\int_0^{2\pi}[4 + 4\sin x + \sin^2 x]dx$
$= \frac{1}{8\pi}\int_0^{2\pi}[4 + 4\sin x]dx + \frac{1}{8\pi}\int_0^{2\pi}[\sin^2 x]dx = \frac{1}{8\pi}\int_0^{2\pi}[4 + 4\sin x]dx + \frac{1}{8\pi}\int_0^{2\pi}\left[\frac{1 - \cos 2x}{2}\right]dx$
$= \frac{1}{8\pi}[4x - 4\cos x +]_0^{2\pi} + \frac{1}{16\pi}\int_0^{2\pi}dx - \frac{1}{16\pi}\int_0^{2\pi}\cos 2x\,dx$ $[u = 2x \Rightarrow du = 2dx, x = 0 \Rightarrow u = 0, x = 2\pi \Rightarrow u = 4\pi]$



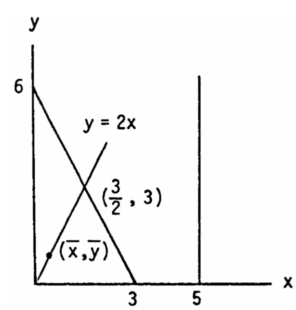
$\rightarrow \frac{1}{8\pi}[4x - 4\cos x]_0^{2\pi} + \frac{1}{16\pi}[x]_0^{2\pi} - \frac{1}{32\pi}\int_0^{4\pi} \cos u\, du = \frac{1}{8\pi}[4x - 4\cos x]_0^{2\pi} + \frac{1}{16\pi}[x]_0^{2\pi} - \frac{1}{32\pi}[\sin u]_0^{4\pi}$
$= \frac{1}{8\pi}(8\pi - 4) - \frac{1}{8\pi}(0 - 4) + \frac{1}{16\pi}(2\pi) - 0 - 0 = \frac{9}{8} \Rightarrow \left(\frac{2\pi - 1}{2}, \frac{9}{8}\right)$ is the center of mass.

31. Consider the curve as an infinite number of line segments joined together. From the derivation of arc length we have that the length of a particular segment is $ds = \sqrt{(dx)^2 + (dy)^2}$. This implies that $M_x = \int \delta y\, ds$, $M_y = \int \delta x\, ds$ and $M = \int \delta\, ds$. If δ is constant, then $\overline{x} = \frac{M_y}{M} = \frac{\int x\, ds}{\int ds} = \frac{\int x\, ds}{\text{length}}$ and $\overline{y} = \frac{M_x}{M} = \frac{\int y\, ds}{\int ds} = \frac{\int y\, ds}{\text{length}}$.

32. Applying the symmetry argument analogous to the one used in Exercise 1, we find that $\overline{x} = 0$. The typical vertical strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{a + \frac{x^2}{4p}}{2}\right)$, length: $a - \frac{x^2}{4p}$, width: dx, area: $dA = \left(a - \frac{x^2}{4p}\right) dx$, mass: $dm = \delta\, dA$
$= \delta \left(a - \frac{x^2}{4p}\right) dx$. Thus, $M_x = \int \widetilde{y}\, dm = \int_{-2\sqrt{pa}}^{2\sqrt{pa}} \frac{1}{2}\left(a + \frac{x^2}{4p}\right)\left(a - \frac{x^2}{4p}\right) \delta\, dx = \frac{\delta}{2}\int_{-2\sqrt{pa}}^{2\sqrt{pa}} \left(a^2 - \frac{x^4}{16p^2}\right) dx$
$= \frac{\delta}{2}\left[a^2 x - \frac{x^5}{80p^2}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2 \cdot \frac{\delta}{2}\left[a^2 x - \frac{x^5}{80p^2}\right]_0^{2\sqrt{pa}} = \delta\left(2a^2\sqrt{pa} - \frac{2^5 p^2 a^2 \sqrt{pa}}{80p^2}\right) = 2a^2\delta\sqrt{pa}\left(1 - \frac{16}{80}\right) = 2a^2\delta\sqrt{pa}\left(\frac{80-16}{80}\right)$
$= 2a^2\delta\sqrt{pa}\left(\frac{64}{80}\right) = \frac{8a^2\delta\sqrt{pa}}{5}$; $M = \int dm = \delta\int_{-2\sqrt{pa}}^{2\sqrt{pa}} \left(a - \frac{x^2}{4p}\right) dx = \delta\left[ax - \frac{x^3}{12p}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2 \cdot \delta\left[ax - \frac{x^3}{12p}\right]_0^{2\sqrt{pa}}$
$= 2\delta\left(2a\sqrt{pa} - \frac{2^3 pa\sqrt{pa}}{12p}\right) = 4a\delta\sqrt{pa}\left(1 - \frac{4}{12}\right) = 4a\delta\sqrt{pa}\left(\frac{12-4}{12}\right) = \frac{8a\delta\sqrt{pa}}{3}$. So $\overline{y} = \frac{M_x}{M} = \left(\frac{8a^2\delta\sqrt{pa}}{5}\right)\left(\frac{3}{8a\delta\sqrt{pa}}\right)$
$= \frac{3}{5}a$, as claimed.

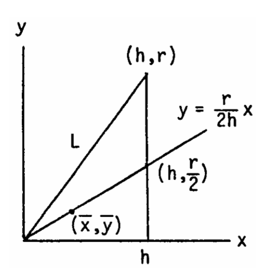
33. The centroid of the square is located at $(2, 2)$. The volume is $V = (2\pi)(\overline{y})(A) = (2\pi)(2)(8) = 32\pi$ and the surface area is $S = (2\pi)(\overline{y})(L) = (2\pi)(2)\left(4\sqrt{8}\right) = 32\sqrt{2}\pi$ (where $\sqrt{8}$ is the length of a side).

34. The midpoint of the hypotenuse of the triangle is $\left(\frac{3}{2}, 3\right)$ $\Rightarrow$ $y = 2x$ is an equation of the median $\Rightarrow$ the line $y = 2x$ contains the centroid. The point $\left(\frac{3}{2}, 3\right)$ is $\frac{3\sqrt{5}}{2}$ units from the origin $\Rightarrow$ the x-coordinate of the centroid solves the equation $\sqrt{\left(x - \frac{3}{2}\right)^2 + (2x - 3)^2}$
$= \frac{\sqrt{5}}{2} \Rightarrow \left(x^2 - 3x + \frac{9}{4}\right) + (4x^2 - 12x + 9) = \frac{5}{4}$
$\Rightarrow 5x^2 - 15x + 9 = -1$
$\Rightarrow x^2 - 3x + 2 = (x - 2)(x - 1) = 0 \Rightarrow \overline{x} = 1$ since the centroid must lie inside the triangle $\Rightarrow \overline{y} = 2$. By the Theorem of Pappus, the volume is $V =$ (distance traveled by the centroid)(area of the region) $= 2\pi(5 - \overline{x})\left[\frac{1}{2}(3)(6)\right]$
$= (2\pi)(4)(9) = 72\pi$



35. The centroid is located at $(2, 0) \Rightarrow V = (2\pi)(\overline{x})(A) = (2\pi)(2)(\pi) = 4\pi^2$

36. We create the cone by revolving the triangle with vertices $(0, 0)$, (h, r) and $(h, 0)$ about the x-axis (see the accompanying figure). Thus, the cone has height h and base radius r. By Theorem of Pappus, the lateral surface area swept out by the hypotenuse L is given by $S = 2\pi\overline{y}L = 2\pi\left(\frac{r}{2}\right)\sqrt{h^2 + r^2}$ $= \pi r\sqrt{r^2 + h^2}$. To calculate the volume we need the position of the centroid of the triangle. From the diagram we see that the centroid lies on the line $y = \frac{r}{2h}x$. The x-coordinate of the centroid solves the equation $\sqrt{(x - h)^2 + \left(\frac{r}{2h}x - \frac{r}{2}\right)^2}$



$= \frac{1}{3}\sqrt{h^2 + \frac{r^2}{4}} \Rightarrow \left(\frac{4h^2+r^2}{4h^2}\right)x^2 - \left(\frac{4h^2+r^2}{2h}\right)x + \frac{r^2}{4} + \frac{2(r^2+4h^2)}{9} = 0 \Rightarrow x = \frac{2h}{3}$ or $\frac{4h}{3} \Rightarrow \overline{x} = \frac{2h}{3}$, since the centroid must lie inside the triangle $\Rightarrow \overline{y} = \frac{r}{2h}\overline{x} = \frac{r}{3}$. By the Theorem of Pappus, $V = \left[2\pi\left(\frac{r}{3}\right)\right]\left(\frac{1}{2}hr\right) = \frac{1}{3}\pi r^2 h$.

37. $S = 2\pi\overline{y}L \Rightarrow 4\pi a^2 = (2\pi\overline{y})(\pi a) \Rightarrow \overline{y} = \frac{2a}{\pi}$, and by symmetry $\overline{x} = 0$

38. $S = 2\pi\rho L \Rightarrow \left[2\pi\left(a - \frac{2a}{\pi}\right)\right](\pi a) = 2\pi a^2(\pi - 2)$

39. $V = 2\pi\overline{y}A \Rightarrow \frac{4}{3}\pi ab^2 = (2\pi\overline{y})\left(\frac{\pi ab}{2}\right) \Rightarrow \overline{y} = \frac{4b}{3\pi}$ and by symmetry $\overline{x} = 0$

40. $V = 2\pi\rho A \Rightarrow V = \left[2\pi\left(a + \frac{4a}{3\pi}\right)\right]\left(\frac{\pi a^2}{2}\right) = \frac{\pi a^3(3\pi+4)}{3}$

41. $V = 2\pi\rho A = (2\pi)$(area of the region) $\cdot$ (distance from the centroid to the line $y = x - a$). We must find the distance from $\left(0, \frac{4a}{3\pi}\right)$ to $y = x - a$. The line containing the centroid and perpendicular to $y = x - a$ has slope -1 and contains the point $\left(0, \frac{4a}{3\pi}\right)$. This line is $y = -x + \frac{4a}{3\pi}$. The intersection of $y = x - a$ and $y = -x + \frac{4a}{3\pi}$ is the point $\left(\frac{4a+3a\pi}{6\pi}, \frac{4a-3a\pi}{6\pi}\right)$. Thus, the distance from the centroid to the line $y = x - a$ is $\sqrt{\left(\frac{4a+3a\pi}{6\pi}\right)^2 + \left(\frac{4a}{3\pi} - \frac{4a}{6\pi} + \frac{3a\pi}{6\pi}\right)^2} = \frac{\sqrt{2}(4a+3a\pi)}{6\pi}$
$\Rightarrow V = (2\pi)\left(\frac{\sqrt{2}(4a+3a\pi)}{6\pi}\right)\left(\frac{\pi a^2}{2}\right) = \frac{\sqrt{2}\pi a^3(4+3\pi)}{6}$

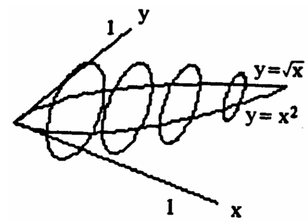
42. The line perpendicular to $y = x - a$ and passing through the centroid $\left(0, \frac{2a}{\pi}\right)$ has equation $y = -x + \frac{2a}{\pi}$. The intersection of the two perpendicular lines occurs when $x - a = -x + \frac{2a}{\pi} \Rightarrow x = \frac{2a+a\pi}{2\pi} \Rightarrow y = \frac{2a-a\pi}{2\pi}$. Thus the distance from the centroid to the line $y = x - a$ is $\sqrt{\left(\frac{2a+\pi a}{2} - 0\right)^2 + \left(\frac{2a-\pi a}{2} - \frac{2a}{2}\right)^2} = \frac{a(2+\pi)}{\sqrt{2}\pi}$. Therefore, by the Theorem of Pappus the surface area is $S = 2\pi\left[\frac{a(2+\pi)}{\sqrt{2}\pi}\right](\pi a) = \sqrt{2}\pi a^2(2+\pi)$.

43. If we revolve the region about the y-axis: $r = a$, $h = b \Rightarrow A = \frac{1}{2}ab$, $V = \frac{1}{3}\pi a^2 b$, and $\rho = \overline{x}$. By the Theorem of Pappus: $\frac{1}{3}\pi a^2 b = 2\pi\overline{x}\left(\frac{1}{2}ab\right) \Rightarrow \overline{x} = \frac{a}{3}$; If we revolve the region about the x-axis: $r = b$, $h = a \Rightarrow A = \frac{1}{2}ab$, $V = \frac{1}{3}\pi b^2 a$, and $\rho = \overline{y}$. By the Theorem of Pappus: $\frac{1}{3}\pi b^2 a = 2\pi\overline{y}\left(\frac{1}{2}ab\right) \Rightarrow \overline{y} = \frac{b}{3} \Rightarrow \left(\frac{a}{3}, \frac{b}{3}\right)$ is the center of mass.

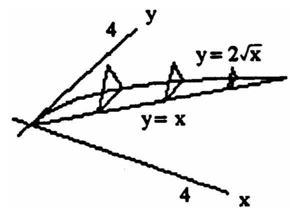
44. Let $O(0, 0)$, $P(a, c)$, and $Q(a, b)$ be the vertices of the given triangle. If we revolve the region about the x-axis: Let R be the point $R(a, 0)$. The volume is given by the volume of the outer cone, radius $= RP = c$, minus the volume of the inner cone, radius $= RQ = b$, thus $V = \frac{1}{3}\pi c^2 a - \frac{1}{3}\pi b^2 a = \frac{1}{3}\pi a(c^2 - b^2)$, the area is given by the area of triangle OPR minus area of triangle OQR, $A = \frac{1}{2}ac - \frac{1}{2}ab = \frac{1}{2}a(c - b)$, and $\rho = \overline{y}$. By the Theorem of Pappus: $\frac{1}{3}\pi a(c^2 - b^2)$
$= 2\pi\overline{y}\left[\frac{1}{2}a(c - b)\right] \Rightarrow \overline{y} = \frac{c+b}{3}$; If we revolve the region about the y-axis: Let S and T be the points $S(0, c)$ and $T(0, b)$, respectively. Then the volume is the volume of the cylinder with radius $OR = a$ and height $RP = c$, minus the sum of the volumes of the cone with radius $= SP = a$ and height $= OS = c$ and the portion of the cylinder with height $= OT = b$ and radius $= TQ = a$ with a cone of height $= OT = b$ and radius $= TQ = a$ removed. Thus
$V = \pi a^2 c - \left[\frac{1}{3}\pi a^2 c + \left(\pi a^2 b - \frac{1}{3}\pi a^2 b\right)\right] = \frac{2}{3}\pi a^2 c - \frac{2}{3}\pi a^2 b = \frac{2}{3}\pi a^2(a - b)$. The area of the triangle is the same as before, $A = \frac{1}{2}ac - \frac{1}{2}ab = \frac{1}{2}a(c - b)$, and $\rho = \overline{x}$. By the Theorem of Pappus: $\frac{2}{3}\pi a^2(a - b) = 2\pi\overline{x}\left[\frac{1}{2}a(c - b)\right]$
$\Rightarrow \overline{x} = \frac{2a(a-b)}{3(c-b)} \Rightarrow \left(\frac{2a(a-b)}{3(c-b)}, \frac{c+b}{2}\right)$ is the center of mass.

CHAPTER 6 PRACTICE EXERCISES

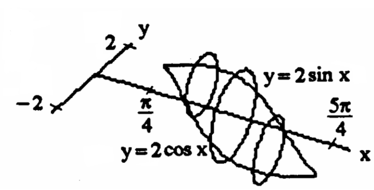
1. $A(x) = \frac{\pi}{4}\,(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{x} - x^2\right)^2$
$= \frac{\pi}{4}\left(x - 2\sqrt{x}\cdot x^2 + x^4\right);\ a = 0,\ b = 1$
$\Rightarrow\ V = \int_a^b A(x)\,dx = \frac{\pi}{4}\int_0^1 \left(x - 2x^{5/2} + x^4\right) dx$
$= \frac{\pi}{4}\left[\frac{x^2}{2} - \frac{4}{7}\,x^{7/2} + \frac{x^5}{5}\right]_0^1 = \frac{\pi}{4}\left(\frac{1}{2} - \frac{4}{7} + \frac{1}{5}\right)$
$= \frac{\pi}{4\cdot 70}\,(35 - 40 + 14) = \frac{9\pi}{280}$



2. $A(x) = \frac{1}{2}\,(\text{side})^2\left(\sin\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}\left(2\sqrt{x} - x\right)^2$
$= \frac{\sqrt{3}}{4}\left(4x - 4x\sqrt{x} + x^2\right);\ a = 0,\ b = 4$
$\Rightarrow\ V = \int_a^b A(x)\,dx = \frac{\sqrt{3}}{4}\int_0^4 \left(4x - 4x^{3/2} + x^2\right) dx$
$= \frac{\sqrt{3}}{4}\left[2x^2 - \frac{8}{5}\,x^{5/2} + \frac{x^3}{3}\right]_0^4 = \frac{\sqrt{3}}{4}\left(32 - \frac{8\cdot 32}{5} + \frac{64}{3}\right)$
$= \frac{32\sqrt{3}}{4}\left(1 - \frac{8}{5} + \frac{2}{3}\right) = \frac{8\sqrt{3}}{15}\,(15 - 24 + 10) = \frac{8\sqrt{3}}{15}$



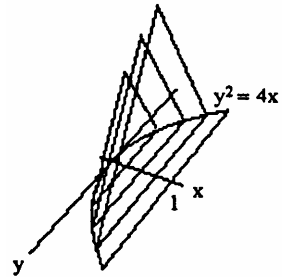
3. $A(x) = \frac{\pi}{4}\,(\text{diameter})^2 = \frac{\pi}{4}\,(2\sin x - 2\cos x)^2$
$= \frac{\pi}{4}\cdot 4\left(\sin^2 x - 2\sin x\cos x + \cos^2 x\right)$
$= \pi(1 - \sin 2x);\ a = \frac{\pi}{4},\ b = \frac{5\pi}{4}$
$\Rightarrow\ V = \int_a^b A(x)\,dx = \pi\int_{\pi/4}^{5\pi/4} (1 - \sin 2x)\,dx$
$= \pi\left[x + \frac{\cos 2x}{2}\right]_{\pi/4}^{5\pi/4}$
$= \pi\left[\left(\frac{5\pi}{4} + \frac{\cos\frac{5\pi}{2}}{2}\right) - \left(\frac{\pi}{4} - \frac{\cos\frac{\pi}{2}}{2}\right)\right] = \pi^2$



4. $A(x) = (\text{edge})^2 = \left(\left(\sqrt{6} - \sqrt{x}\right)^2 - 0\right)^2 = \left(\sqrt{6} - \sqrt{x}\right)^4 = 36 - 24\sqrt{6}\,\sqrt{x} + 36x - 4\sqrt{6}\,x^{3/2} + x^2;$
$a = 0,\ b = 6\ \Rightarrow\ V = \int_a^b A(x)\,dx = \int_0^6 \left(36 - 24\sqrt{6}\,\sqrt{x} + 36x - 4\sqrt{6}\,x^{3/2} + x^2\right) dx$
$= \left[36x - 24\sqrt{6}\cdot\frac{2}{3}\,x^{3/2} + 18x^2 - 4\sqrt{6}\cdot\frac{2}{5}\,x^{5/2} + \frac{x^3}{3}\right]_0^6 = 216 - 16\cdot\sqrt{6}\,\sqrt{6}\cdot 6 + 18\cdot 6^2 - \frac{8}{5}\,\sqrt{6}\,\sqrt{6}\cdot 6^2 + \frac{6^3}{3}$
$= 216 - 576 + 648 - \frac{1728}{5} + 72 = 360 - \frac{1728}{5} = \frac{1800 - 1728}{5} = \frac{72}{5}$

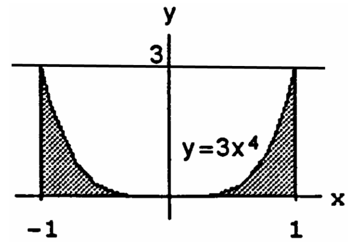
5. $A(x) = \frac{\pi}{4}\,(\text{diameter})^2 = \frac{\pi}{4}\left(2\sqrt{x} - \frac{x^2}{4}\right)^2 = \frac{\pi}{4}\left(4x - x^{5/2} + \frac{x^4}{16}\right);\ a = 0,\ b = 4\ \Rightarrow\ V = \int_a^b A(x)\,dx$
$= \frac{\pi}{4}\int_0^4 \left(4x - x^{5/2} + \frac{x^4}{16}\right) dx = \frac{\pi}{4}\left[2x^2 - \frac{2}{7}\,x^{7/2} + \frac{x^5}{5\cdot 16}\right]_0^4 = \frac{\pi}{4}\left(32 - 32\cdot\frac{8}{7} + \frac{2}{5}\cdot 32\right)$
$= \frac{32\pi}{4}\left(1 - \frac{8}{7} + \frac{2}{5}\right) = \frac{8\pi}{35}\,(35 - 40 + 14) = \frac{72\pi}{35}$

6. $A(x) = \frac{1}{2}\,(\text{edge})^2 \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}\left[2\sqrt{x} - \left(-2\sqrt{x}\right)\right]^2$
$= \frac{\sqrt{3}}{4}\left(4\sqrt{x}\right)^2 = 4\sqrt{3}\,x;\ a = 0,\ b = 1$
$\Rightarrow\ V = \int_a^b A(x)\,dx = \int_0^1 4\sqrt{3}\,x\,dx = \left[2\sqrt{3}\,x^2\right]_0^1$
$= 2\sqrt{3}$



7. (a) *disk method*:

$V = \int_a^b \pi R^2(x)\,dx = \int_{-1}^{1} \pi\left(3x^4\right)^2 dx = \pi \int_{-1}^{1} 9x^8\,dx$
$= \pi\left[x^9\right]_{-1}^{1} = 2\pi$



(b) *shell method*:

$V = \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dx = \int_0^1 2\pi x\left(3x^4\right) dx = 2\pi \cdot 3\int_0^1 x^5\,dx = 2\pi \cdot 3\left[\frac{x^6}{6}\right]_0^1 = \pi$

Note: The lower limit of integration is 0 rather than -1.

(c) *shell method*:

$V = \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dx = 2\pi\int_{-1}^{1}(1-x)\left(3x^4\right) dx = 2\pi\left[\frac{3x^5}{5} - \frac{x^6}{2}\right]_{-1}^{1} = 2\pi\left[\left(\frac{3}{5}-\frac{1}{2}\right) - \left(-\frac{3}{5}-\frac{1}{2}\right)\right] = \frac{12\pi}{5}$

(d) *washer method*:

$R(x) = 3,\ r(x) = 3 - 3x^4 = 3\left(1-x^4\right) \Rightarrow V = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx = \int_{-1}^{1} \pi\left[9 - 9\left(1-x^4\right)^2\right] dx$
$= 9\pi\int_{-1}^{1}\left[1 - \left(1 - 2x^4 + x^8\right)\right] dx = 9\pi\int_{-1}^{1}\left(2x^4 - x^8\right) dx = 9\pi\left[\frac{2x^5}{5} - \frac{x^9}{9}\right]_{-1}^{1} = 18\pi\left[\frac{2}{5} - \frac{1}{9}\right] = \frac{2\pi\cdot 13}{5} = \frac{26\pi}{5}$

8. (a) *washer method*:

$R(x) = \frac{4}{x^3},\ r(x) = \frac{1}{2} \Rightarrow V = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx = \int_1^2 \pi\left[\left(\frac{4}{x^3}\right)^2 - \left(\frac{1}{2}\right)^2\right] dx = \pi\left[-\frac{16}{5}x^{-5} - \frac{x}{4}\right]_1^2$
$= \pi\left[\left(\frac{-16}{5\cdot 32} - \frac{1}{2}\right) - \left(-\frac{16}{5} - \frac{1}{4}\right)\right] = \pi\left(-\frac{1}{10} - \frac{1}{2} + \frac{16}{5} + \frac{1}{4}\right) = \frac{\pi}{20}(-2 - 10 + 64 + 5) = \frac{57\pi}{20}$

(b) *shell method*:

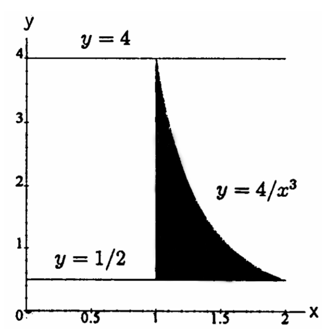
$V = 2\pi\int_1^2 x\left(\frac{4}{x^3} - \frac{1}{2}\right) dx = 2\pi\left[-4x^{-1} - \frac{x^2}{4}\right]_1^2 = 2\pi\left[\left(-\frac{4}{2} - 1\right) - \left(-4 - \frac{1}{4}\right)\right] = 2\pi\left(\frac{5}{4}\right) = \frac{5\pi}{2}$

(c) *shell method*:

$V = 2\pi\int_a^b \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dx = 2\pi\int_1^2 (2-x)\left(\frac{4}{x^3} - \frac{1}{2}\right) dx = 2\pi\int_1^2\left(\frac{8}{x^3} - \frac{4}{x^2} - 1 + \frac{x}{2}\right) dx$
$= 2\pi\left[-\frac{4}{x^2} + \frac{4}{x} - x + \frac{x^2}{4}\right]_1^2 = 2\pi\left[(-1 + 2 - 2 + 1) - \left(-4 + 4 - 1 + \frac{1}{4}\right)\right] = \frac{3\pi}{2}$

(d) *washer method*:

$V = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx$
$= \pi\int_1^2\left[\left(\frac{7}{2}\right)^2 - \left(4 - \frac{4}{x^3}\right)^2\right] dx$
$= \frac{49\pi}{4} - 16\pi\int_1^2\left(1 - 2x^{-3} + x^{-6}\right) dx$
$= \frac{49\pi}{4} - 16\pi\left[x + x^{-2} - \frac{x^{-5}}{5}\right]_1^2$
$= \frac{49\pi}{4} - 16\pi\left[\left(2 + \frac{1}{4} - \frac{1}{5\cdot 32}\right) - \left(1 + 1 - \frac{1}{5}\right)\right]$
$= \frac{49\pi}{4} - 16\pi\left(\frac{1}{4} - \frac{1}{160} + \frac{1}{5}\right)$
$= \frac{49\pi}{4} - \frac{16\pi}{160}(40 - 1 + 32) = \frac{49\pi}{4} - \frac{71\pi}{10} = \frac{103\pi}{20}$



9. (a) *disk method*:

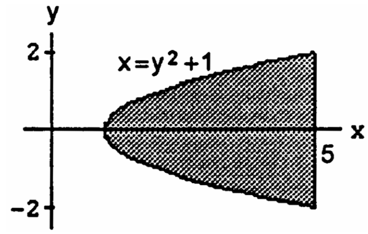
$V = \pi\int_1^5\left(\sqrt{x-1}\right)^2 dx = \pi\int_1^5 (x-1)\,dx = \pi\left[\frac{x^2}{2} - x\right]_1^5$
$= \pi\left[\left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)\right] = \pi\left(\frac{24}{2} - 4\right) = 8\pi$

(b) *washer method*:

$R(y) = 5,\ r(y) = y^2 + 1 \Rightarrow V = \int_c^d \pi\left[R^2(y) - r^2(y)\right] dy = \pi\int_{-2}^{2}\left[25 - \left(y^2+1\right)^2\right] dy$
$= \pi\int_{-2}^{2}\left(25 - y^4 - 2y^2 - 1\right) dy = \pi\int_{-2}^{2}\left(24 - y^4 - 2y^2\right) dy = \pi\left[24y - \frac{y^5}{5} - \frac{2}{3}y^3\right]_{-2}^{2} = 2\pi\left(24\cdot 2 - \frac{32}{5} - \frac{2}{3}\cdot 8\right)$

$= 32\pi\left(3 - \frac{2}{5} - \frac{1}{3}\right) = \frac{32\pi}{15}(45 - 6 - 5) = \frac{1088\pi}{15}$

(c) *disk method*:



$R(y) = 5 - (y^2 + 1) = 4 - y^2$

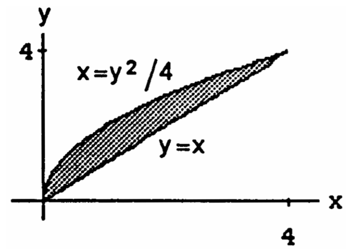
$\Rightarrow\ V = \int_c^d \pi R^2(y)\,dy = \int_{-2}^{2} \pi\left(4 - y^2\right)^2 dy$

$= \pi \int_{-2}^{2} (16 - 8y^2 + y^4)\,dy$

$= \pi\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_{-2}^{2} = 2\pi\left(32 - \frac{64}{3} + \frac{32}{5}\right)$

$= 64\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64\pi}{15}(15 - 10 + 3) = \frac{512\pi}{15}$

10. (a) *shell method*:



$V = \int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^4 2\pi y\left(y - \frac{y^2}{4}\right) dy$

$= 2\pi \int_0^4 \left(y^2 - \frac{y^3}{4}\right) dy = 2\pi\left[\frac{y^3}{3} - \frac{y^4}{16}\right]_0^4 = 2\pi\left(\frac{64}{3} - \frac{64}{4}\right)$

$= \frac{2\pi}{12} \cdot 64 = \frac{32\pi}{3}$

(b) *shell method*:

$V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^4 2\pi x\left(2\sqrt{x} - x\right) dx = 2\pi \int_0^4 \left(2x^{3/2} - x^2\right) dx = 2\pi\left[\frac{4}{5}x^{5/2} - \frac{x^3}{3}\right]_0^4$

$= 2\pi\left(\frac{4}{5} \cdot 32 - \frac{64}{3}\right) = \frac{128\pi}{15}$

(c) *shell method*:

$V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^4 2\pi(4 - x)\left(2\sqrt{x} - x\right) dx = 2\pi \int_0^4 \left(8x^{1/2} - 4x - 2x^{3/2} + x^2\right) dx$

$= 2\pi\left[\frac{16}{3}x^{3/2} - 2x^2 - \frac{4}{5}x^{5/2} + \frac{x^3}{3}\right]_0^4 = 2\pi\left(\frac{16}{3} \cdot 8 - 32 - \frac{4}{5} \cdot 32 + \frac{64}{3}\right) = 64\pi\left(\frac{4}{3} - 1 - \frac{4}{5} + \frac{2}{3}\right)$

$= 64\pi\left(1 - \frac{4}{5}\right) = \frac{64\pi}{5}$

(d) *shell method*:

$V = \int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^4 2\pi(4 - y)\left(y - \frac{y^2}{4}\right) dy = 2\pi \int_0^4 \left(4y - y^2 - y^2 + \frac{y^3}{4}\right) dy$

$= 2\pi \int_0^4 \left(4y - 2y^2 + \frac{y^3}{4}\right) dy = 2\pi\left[2y^2 - \frac{2}{3}y^3 + \frac{y^4}{16}\right]_0^4 = 2\pi\left(32 - \frac{2}{3} \cdot 64 + 16\right) = 32\pi\left(2 - \frac{8}{3} + 1\right) = \frac{32\pi}{3}$

11. *disk method*:

$R(x) = \tan x,\ a = 0,\ b = \frac{\pi}{3} \ \Rightarrow\ V = \pi \int_0^{\pi/3} \tan^2 x\,dx = \pi \int_0^{\pi/3} (\sec^2 x - 1)\,dx = \pi[\tan x - x]_0^{\pi/3} = \frac{\pi\left(3\sqrt{3} - \pi\right)}{3}$

12. *disk method*:

$V = \pi \int_0^{\pi} (2 - \sin x)^2\,dx = \pi \int_0^{\pi} (4 - 4\sin x + \sin^2 x)\,dx = \pi \int_0^{\pi} \left(4 - 4\sin x + \frac{1 - \cos 2x}{2}\right) dx$

$= \pi\left[4x + 4\cos x + \frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi} = \pi\left[\left(4\pi - 4 + \frac{\pi}{2} - 0\right) - (0 + 4 + 0 - 0)\right] = \pi\left(\frac{9\pi}{2} - 8\right) = \frac{\pi}{2}(9\pi - 16)$

13. (a) *disk method*:

$V = \pi \int_0^2 \left(x^2 - 2x\right)^2 dx = \pi \int_0^2 (x^4 - 4x^3 + 4x^2)\,dx = \pi\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2 = \pi\left(\frac{32}{5} - 16 + \frac{32}{3}\right)$

$= \frac{16\pi}{15}(6 - 15 + 10) = \frac{16\pi}{15}$

(b) *washer method*:

$V = \int_0^2 \pi\left[1^2 - \left(x^2 - 2x + 1\right)^2\right] dx = \int_0^2 \pi\,dx - \int_0^2 \pi(x - 1)^4\,dx = 2\pi - \left[\pi\,\frac{(x-1)^5}{5}\right]_0^2 = 2\pi - \pi \cdot \frac{2}{5} = \frac{8\pi}{5}$

(c) *shell method*:

$V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = 2\pi \int_0^2 (2 - x)\left[-(x^2 - 2x)\right] dx = 2\pi \int_0^2 (2 - x)(2x - x^2)\,dx$

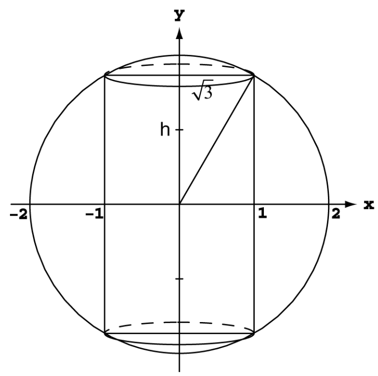
$= 2\pi\int_0^2 (4x - 2x^2 - 2x^2 + x^3)\,dx = 2\pi\int_0^2 (x^3 - 4x^2 + 4x)\,dx = 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 2\pi\left(4 - \frac{32}{3} + 8\right)$
$= \frac{2\pi}{3}(36 - 32) = \frac{8\pi}{3}$

(d) *washer method*:

$V = \pi\int_0^2 [2 - (x^2 - 2x)]^2\,dx - \pi\int_0^2 2^2\,dx = \pi\int_0^2 \left[4 - 4(x^2 - 2x) + (x^2 - 2x)^2\right]dx - 8\pi$
$= \pi\int_0^2 (4 - 4x^2 + 8x + x^4 - 4x^3 + 4x^2)\,dx - 8\pi = \pi\int_0^2 (x^4 - 4x^3 + 8x + 4)\,dx - 8\pi$
$= \pi\left[\frac{x^5}{5} - x^4 + 4x^2 + 4x\right]_0^2 - 8\pi = \pi\left(\frac{32}{5} - 16 + 16 + 8\right) - 8\pi = \frac{\pi}{5}(32 + 40) - 8\pi = \frac{72\pi}{5} - \frac{40\pi}{5} = \frac{32\pi}{5}$

14. *disk method*:
$V = 2\pi\int_0^{\pi/4} 4\tan^2 x\,dx = 8\pi\int_0^{\pi/4} (\sec^2 x - 1)\,dx = 8\pi[\tan x - x]_0^{\pi/4} = 2\pi(4 - \pi)$

15. The material removed from the sphere consists of a cylinder and two "caps." From the diagram, the height of the cylinder is 2h, where $h^2 + \left(\sqrt{3}\right)^2 = 2^2$, i.e. $h = 1$. Thus $V_{cyl} = (2h)\pi\left(\sqrt{3}\right)^2 = 6\pi\ \text{ft}^3$. To get the volume of a cap, use the disk method and $x^2 + y^2 = 2^2$: $V_{cap} = \int_1^2 \pi x^2 dy$
$= \int_1^2 \pi(4 - y^2)dy = \pi\left[4y - \frac{y^3}{3}\right]_1^2$
$= \pi\left[\left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)\right] = \frac{5\pi}{3}\ \text{ft}^3$. Therefore,
$V_{removed} = V_{cyl} + 2V_{cap} = 6\pi + \frac{10\pi}{3} = \frac{28\pi}{3}\ \text{ft}^3$.



16. We rotate the region enclosed by the curve $y = \sqrt{12\left(1 - \frac{4x^2}{121}\right)}$ and the x-axis around the x-axis. To find the volume we use the *disk* method: $V = \int_a^b \pi R^2(x)\,dx = \int_{-11/2}^{11/2} \pi\left(\sqrt{12\left(1 - \frac{4x^2}{121}\right)}\right)^2 dx = \pi\int_{-11/2}^{11/2} 12\left(1 - \frac{4x^2}{121}\right)dx$
$= 12\pi\int_{-11/2}^{11/2}\left(1 - \frac{4x^2}{121}\right)dx = 12\pi\left[x - \frac{4x^3}{363}\right]_{-11/2}^{11/2} = 24\pi\left[\frac{11}{2} - \left(\frac{4}{363}\right)\left(\frac{11}{2}\right)^3\right] = 132\pi\left[1 - \left(\frac{4}{363}\right)\left(\frac{11^2}{4}\right)\right]$
$= 132\pi\left(1 - \frac{1}{3}\right) = \frac{264\pi}{3} = 88\pi \approx 276\ \text{in}^3$

17. $y = x^{1/2} - \frac{x^{3/2}}{3} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(\frac{1}{x} - 2 + x\right) \Rightarrow L = \int_1^4 \sqrt{1 + \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)}\,dx$
$\Rightarrow L = \int_1^4 \sqrt{\frac{1}{4}\left(\frac{1}{x} + 2 + x\right)}\,dx = \int_1^4 \sqrt{\frac{1}{4}\left(x^{-1/2} + x^{1/2}\right)^2}\,dx = \int_1^4 \frac{1}{2}\left(x^{-1/2} + x^{1/2}\right)dx = \frac{1}{2}\left[2x^{1/2} + \frac{2}{3}x^{3/2}\right]_1^4$
$= \frac{1}{2}\left[\left(4 + \frac{2}{3}\cdot 8\right) - \left(2 + \frac{2}{3}\right)\right] = \frac{1}{2}\left(2 + \frac{14}{3}\right) = \frac{10}{3}$

18. $x = y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3}y^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4y^{-2/3}}{9} \Rightarrow L = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_1^8 \sqrt{1 + \frac{4}{9y^{2/3}}}\,dy$
$= \int_1^8 \frac{\sqrt{9y^{2/3} + 4}}{3y^{1/3}}\,dy = \frac{1}{3}\int_1^8 \sqrt{9y^{2/3} + 4}\left(y^{-1/3}\right)dy$; $\left[u = 9y^{2/3} + 4 \Rightarrow du = 6y^{-1/3}\,dy;\ y = 1 \Rightarrow u = 13,\right.$
$\left. y = 8 \Rightarrow u = 40\right] \rightarrow L = \frac{1}{18}\int_{13}^{40} u^{1/2}\,du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.634$

19. $y = \frac{5}{12}x^{6/5} - \frac{5}{8}x^{4/5} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{1/5} - \frac{1}{2}x^{-1/5} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)$
$\Rightarrow L = \int_1^{32} \sqrt{1 + \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)}\,dx \Rightarrow L = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{2/5} + 2 + x^{-2/5}\right)}\,dx = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{1/5} + x^{-1/5}\right)^2}\,dx$

$= \int_1^{32} \frac{1}{2}\left(x^{1/5} + x^{-1/5}\right) dx = \frac{1}{2}\left[\frac{5}{6} x^{6/5} + \frac{5}{4} x^{4/5}\right]_1^{32} = \frac{1}{2}\left[\left(\frac{5}{6} \cdot 2^6 + \frac{5}{4} \cdot 2^4\right) - \left(\frac{5}{6} + \frac{5}{4}\right)\right] = \frac{1}{2}\left(\frac{315}{6} + \frac{75}{4}\right)$
$= \frac{1}{48}(1260 + 450) = \frac{1710}{48} = \frac{285}{8}$

20. $x = \frac{1}{12} y^3 + \frac{1}{y} \Rightarrow \frac{dx}{dy} = \frac{1}{4} y^2 - \frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{16} y^4 - \frac{1}{2} + \frac{1}{y^4} \Rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{1}{16} y^4 - \frac{1}{2} + \frac{1}{y^4}\right)}\, dy$
$= \int_1^2 \sqrt{\frac{1}{16} y^4 + \frac{1}{2} + \frac{1}{y^4}}\, dy = \int_1^2 \sqrt{\left(\frac{1}{4} y^2 + \frac{1}{y^2}\right)^2}\, dy = \int_1^2 \left(\frac{1}{4} y^2 + \frac{1}{y^2}\right) dy = \left[\frac{1}{12} y^3 - \frac{1}{y}\right]_1^2$
$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) = \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$

21. $S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$; $\frac{dy}{dx} = \frac{1}{\sqrt{2x+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{2x+1} \Rightarrow S = \int_0^3 2\pi \sqrt{2x+1} \sqrt{1 + \frac{1}{2x+1}}\, dx$
$= 2\pi \int_0^3 \sqrt{2x+1} \sqrt{\frac{2x+2}{2x+1}}\, dx = 2\sqrt{2}\pi \int_0^3 \sqrt{x+1}\, dx = 2\sqrt{2}\pi \left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = 2\sqrt{2}\pi \cdot \frac{2}{3}(8-1) = \frac{28\pi\sqrt{2}}{3}$

22. $S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$; $\frac{dy}{dx} = x^2 \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^4 \Rightarrow S = \int_0^1 2\pi \cdot \frac{x^3}{3} \sqrt{1 + x^4}\, dx = \frac{\pi}{6} \int_0^1 \sqrt{1 + x^4}\, (4x^3)\, dx$
$= \frac{\pi}{6} \int_0^1 \sqrt{1 + x^4}\, d(1 + x^4) = \frac{\pi}{6}\left[\frac{2}{3}(1 + x^4)^{3/2}\right]_0^1 = \frac{\pi}{9}\left[2\sqrt{2} - 1\right]$

23. $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$; $\frac{dx}{dy} = \frac{\left(\frac{1}{2}\right)(4 - 2y)}{\sqrt{4y - y^2}} = \frac{2 - y}{\sqrt{4y - y^2}} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = \frac{4y - y^2 + 4 - 4y + y^2}{4y - y^2} = \frac{4}{4y - y^2}$
$\Rightarrow S = \int_1^2 2\pi \sqrt{4y - y^2} \sqrt{\frac{4}{4y - y^2}}\, dy = 4\pi \int_1^2 dx = 4\pi$

24. $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$; $\frac{dx}{dy} = \frac{1}{2\sqrt{y}} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4y} = \frac{4y+1}{4y} \Rightarrow S = \int_2^6 2\pi \sqrt{y} \cdot \frac{\sqrt{4y+1}}{\sqrt{4y}}\, dy$
$= \pi \int_2^6 \sqrt{4y+1}\, dy = \frac{\pi}{4}\left[\frac{2}{3}(4y+1)^{3/2}\right]_2^6 = \frac{\pi}{6}(125 - 27) = \frac{\pi}{6}(98) = \frac{49\pi}{3}$

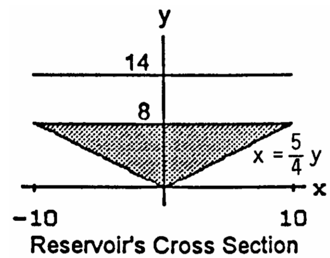
25. The equipment alone: the force required to lift the equipment is equal to its weight $\Rightarrow F_1(x) = 100$ N. The work done is $W_1 = \int_a^b F_1(x)\, dx = \int_0^{40} 100\, dx = [100x]_0^{40} = 4000$ J; the rope alone: the force required to lift the rope is equal to the weight of the rope paid out at elevation x $\Rightarrow F_2(x) = 0.8(40 - x)$. The work done is $W_2 = \int_a^b F_2(x)\, dx = \int_0^{40} 0.8(40 - x)\, dx = 0.8\left[40x - \frac{x^2}{2}\right]_0^{40} = 0.8\left(40^2 - \frac{40^2}{2}\right) = \frac{(0.8)(1600)}{2} = 640$ J; the total work is $W = W_1 + W_2 = 4000 + 640 = 4640$ J

26. The force required to lift the water is equal to the water's weight, which varies steadily from $8 \cdot 800$ lb to $8 \cdot 400$ lb over the 4750 ft elevation. When the truck is x ft off the base of Mt. Washington, the water weight is $F(x) = 8 \cdot 800 \cdot \left(\frac{2 \cdot 4750 - x}{2 \cdot 4750}\right) = (6400)\left(1 - \frac{x}{9500}\right)$ lb. The work done is $W = \int_a^b F(x)\, dx$
$= \int_0^{4750} 6400\left(1 - \frac{x}{9500}\right) dx = 6400\left[x - \frac{x^2}{2 \cdot 9500}\right]_0^{4750} = 6400\left(4750 - \frac{4750^2}{4 \cdot 4750}\right) = \left(\frac{3}{4}\right)(6400)(4750)$
$= 22{,}800{,}000$ ft · lb

27. Force constant: $F = kx \Rightarrow 20 = k \cdot 1 \Rightarrow k = 20$ lb/ft; the work to stretch the spring 1 ft is
$W = \int_0^1 kx\, dx = k \int_0^1 x\, dx = \left[20 \frac{x^2}{2}\right]_0^1 = 10$ ft · lb; the work to stretch the spring an additional foot is
$W = \int_1^2 kx\, dx = k \int_1^2 x\, dx = 20\left[\frac{x^2}{2}\right]_1^2 = 20\left(\frac{4}{2} - \frac{1}{2}\right) = 20\left(\frac{3}{2}\right) = 30$ ft · lb

28. Force constant: $F = kx \Rightarrow 200 = k(0.8) \Rightarrow k = 250$ N/m; the 300 N force stretches the spring $x = \frac{F}{k}$ $= \frac{300}{250} = 1.2$ m; the work required to stretch the spring that far is then $W = \int_0^{1.2} F(x)\,dx = \int_0^{1.2} 250x\,dx$ $= [125x^2]_0^{1.2} = 125(1.2)^2 = 180$ J

29. We imagine the water divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 8]$. The typical slab between the planes at y and $y + \Delta y$ has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\frac{5}{4}y\right)^2 \Delta y = \frac{25\pi}{16} y^2\,\Delta y$ ft^3. The force F(y) required to lift this slab is equal to its weight: $F(y) = 62.4\,\Delta V$ $= \frac{(62.4)(25)}{16}\pi y^2\,\Delta y$ lb. The distance through which F(y) must act to lift this slab to the level 6 ft above the top is about $(6 + 8 - y)$ ft, so the work done lifting the slab is about $\Delta W = \frac{(62.4)(25)}{16}\pi y^2(14 - y)\,\Delta y$ ft · lb. The work done lifting all the slabs from $y = 0$ to $y = 8$ to the level 6 ft above the top is approximately



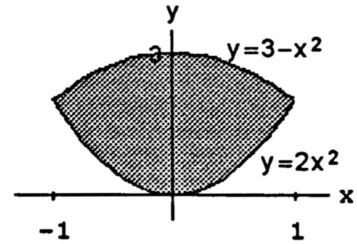
$W \approx \sum_0^8 \frac{(62.4)(25)}{16}\pi y^2\,(14 - y)\,\Delta y$ ft · lb so the work to pump the water is the limit of these Riemann sums as the norm of the partition goes to zero: $W = \int_0^8 \frac{(62.4)(25)}{(16)}\pi y^2(14 - y)\,dy = \frac{(62.4)(25)\pi}{16}\int_0^8 (14y^2 - y^3)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\left[\frac{14}{3}y^3 - \frac{y^4}{4}\right]_0^8$ $= (62.4)\left(\frac{25\pi}{16}\right)\left(\frac{14}{3}\cdot 8^3 - \frac{8^4}{4}\right) \approx 418{,}208.81$ ft · lb

30. The same as in Exercise 29, but change the distance through which F(y) must act to $(8 - y)$ rather than $(6 + 8 - y)$. Also change the upper limit of integration from 8 to 5. The integral is:$W = \int_0^5 \frac{(62.4)(25)\pi}{16} y^2(8 - y)\,dy$ $= (62.4)\left(\frac{25\pi}{16}\right)\int_0^5 (8y^2 - y^3)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\left[\frac{8}{3}y^3 - \frac{y^4}{4}\right]_0^5 = (62.4)\left(\frac{25\pi}{16}\right)\left(\frac{8}{3}\cdot 5^3 - \frac{5^4}{4}\right) \approx 54{,}241.56$ ft · lb

31. The tank's cross section looks like the figure in Exercise 29 with right edge given by $x = \frac{5}{10}y = \frac{y}{2}$. A typical horizontal slab has volume $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{y}{2}\right)^2\Delta y = \frac{\pi}{4}y^2\,\Delta y$. The force required to lift thisslab is its weight: $F(y) = 60 \cdot \frac{\pi}{4}y^2\,\Delta y$. The distance through which F(y) must act is $(2 + 10 - y)$ ft, so the work to pump the liquid is $W = 60\int_0^{10}\pi(12 - y)\left(\frac{y^2}{4}\right)dy = 15\pi\left[\frac{12y^3}{3} - \frac{y^4}{4}\right]_0^{10} = 22{,}500\pi$ ft · lb; the time needed to empty the tank is $\frac{22{,}500\pi \text{ ft·lb}}{275 \text{ ft·lb/sec}} \approx 257$ sec

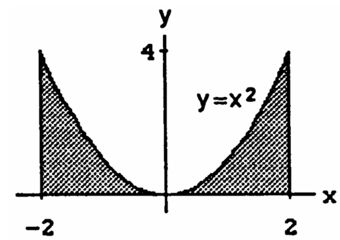
32. A typical horizontal slab has volume about $\Delta V = (20)(2x)\Delta y = (20)\left(2\sqrt{16 - y^2}\right)\Delta y$ and the force required to lift this slab is its weight $F(y) = (57)(20)\left(2\sqrt{16 - y^2}\right)\Delta y$. The distance through which F(y) must act is $(6 + 4 - y)$ ft, so the work to pump the olive oil from the half-full tank is $W = 57\int_{-4}^0 (10 - y)(20)\left(2\sqrt{16 - y^2}\right)dy$ $= 2880\int_{-4}^0 10\sqrt{16 - y^2}\,dy + 1140\int_{-4}^0 (16 - y^2)^{1/2}(-2y)\,dy$ $= 22{,}800 \cdot (\text{area of a quarter circle having radius 4}) + \frac{2}{3}(1140)\left[(16 - y^2)^{3/2}\right]_{-4}^0 = (22{,}800)(4\pi) + 48{,}640$ $= 335{,}153.25$ ft · lb

33. Intersection points: $3 - x^2 = 2x^2 \Rightarrow 3x^2 - 3 = 0$ $\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = -1$ or $x = 1$. Symmetry suggests that $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{2x^2 + (3 - x^2)}{2}\right) = \left(x, \frac{x^2+3}{2}\right)$, length: $(3 - x^2) - 2x^2 = 3(1 - x^2)$, width: dx, area: $dA = 3(1 - x^2)\,dx$, and mass: $dm = \delta \cdot dA$ $= 3\delta(1 - x^2)\,dx \Rightarrow$ the moment about the x-axis is

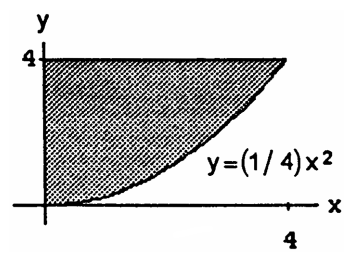


$\tilde{y}\,dm = \frac{3}{2}\delta(x^2+3)(1-x^2)\,dx = \frac{3}{2}\delta(-x^4 - 2x^2 + 3)\,dx \Rightarrow M_x = \int \tilde{y}\,dm = \frac{3}{2}\delta\int_{-1}^{1}(-x^4 - 2x^2 + 3)\,dx$ $= \frac{3}{2}\delta\left[-\frac{x^5}{5} - \frac{2x^3}{3} + 3x\right]_{-1}^{1} = 3\delta\left(-\frac{1}{5} - \frac{2}{3} + 3\right) = \frac{3\delta}{15}(-3 - 10 + 45) = \frac{32\delta}{5}$; $M = \int dm = 3\delta\int_{-1}^{1}(1 - x^2)\,dx$ $= 3\delta\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 6\delta\left(1 - \frac{1}{3}\right) = 4\delta \Rightarrow \bar{y} = \frac{M_x}{M} = \frac{32\delta}{5 \cdot 4\delta} = \frac{8}{5}$. Therefore, the centroid is $(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right)$.

34. Symmetry suggests that $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{x^2}{2}\right)$, length: x^2, width: dx, area: $dA = x^2\,dx$, mass: $dm = \delta \cdot dA = \delta x^2\,dx$ $\Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \frac{\delta}{2}x^2 \cdot x^2\,dx$ $= \frac{\delta}{2}x^4\,dx \Rightarrow M_x = \int \tilde{y}\,dm = \frac{\delta}{2}\int_{-2}^{2} x^4\,dx = \frac{\delta}{10}\left[x^5\right]_{-2}^{2}$

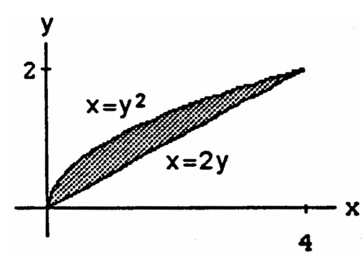


35. The typical *vertical* strip has: center of mass: $(\tilde{x}, \tilde{y})$ $= \left(x, \frac{4 + \frac{x^2}{4}}{2}\right)$, length: $4 - \frac{x^2}{4}$, width: dx, area: $dA = \left(4 - \frac{x^2}{4}\right)dx$, mass: $dm = \delta \cdot dA$ $= \delta\left(4 - \frac{x^2}{4}\right)dx \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \delta \cdot \frac{\left(4 + \frac{x^2}{4}\right)}{2}\left(4 - \frac{x^2}{4}\right)dx = \frac{\delta}{2}\left(16 - \frac{x^4}{16}\right)dx$; the



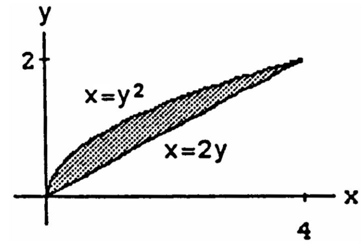
moment about the y-axis is $\tilde{x}\,dm = \delta\left(4 - \frac{x^2}{4}\right) \cdot x\,dx = \delta\left(4x - \frac{x^3}{4}\right)dx$. Thus, $M_x = \int \tilde{y}\,dm = \frac{\delta}{2}\int_0^4\left(16 - \frac{x^4}{16}\right)dx$ $= \frac{\delta}{2}\left[16x - \frac{x^5}{5 \cdot 16}\right]_0^4 = \frac{\delta}{2}\left[64 - \frac{64}{5}\right] = \frac{128\delta}{5}$; $M_y = \int \tilde{x}\,dm = \delta\int_0^4\left(4x - \frac{x^3}{4}\right)dx = \delta\left[2x^2 - \frac{x^4}{16}\right]_0^4$ $= \delta(32 - 16) = 16\delta$; $M = \int dm = \delta\int_0^4\left(4 - \frac{x^2}{4}\right)dx = \delta\left[4x - \frac{x^3}{12}\right]_0^4 = \delta\left(16 - \frac{64}{12}\right) = \frac{32\delta}{3}$ $\Rightarrow \bar{x} = \frac{M_y}{M} = \frac{16 \cdot \delta \cdot 3}{32 \cdot \delta} = \frac{3}{2}$ and $\bar{y} = \frac{M_x}{M} = \frac{128 \cdot \delta \cdot 3}{5 \cdot 32 \cdot \delta} = \frac{12}{5}$. Therefore, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{12}{5}\right)$.

36. A typical *horizontal* strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{y^2 + 2y}{2}, y\right)$, length: $2y - y^2$, width: dy, area: $dA = (2y - y^2)\,dy$, mass: $dm = \delta \cdot dA$ $= \delta(2y - y^2)\,dy$; the moment about the x-axis is $\tilde{y}\,dm = \delta \cdot y \cdot (2y - y^2)\,dy = \delta(2y^2 - y^3)$; the moment about the y-axis is $\tilde{x}\,dm = \delta \cdot \frac{(y^2 + 2y)}{2} \cdot (2y - y^2)\,dy$



$= \frac{\delta}{2}(4y^2 - y^4)\,dy \Rightarrow M_x = \int \tilde{y}\,dm = \delta\int_0^2(2y^2 - y^3)\,dy$ $= \delta\left[\frac{2}{3}y^3 - \frac{y^4}{4}\right]_0^2 = \delta\left(\frac{2}{3} \cdot 8 - \frac{16}{4}\right) = \delta\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{\delta \cdot 16}{12} = \frac{4\delta}{3}$; $M_y = \int \tilde{x}\,dm = \frac{\delta}{2}\int_0^2(4y^2 - y^4)\,dy = \frac{\delta}{2}\left[\frac{4}{3}y^3 - \frac{y^5}{5}\right]_0^2$ $= \frac{\delta}{2}\left(\frac{4 \cdot 8}{3} - \frac{32}{5}\right) = \frac{32\delta}{15}$; $M = \int dm = \delta\int_0^2(2y - y^2)\,dy = \delta\left[y^2 - \frac{y^3}{3}\right]_0^2 = \delta\left(4 - \frac{8}{3}\right) = \frac{4\delta}{3} \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\delta \cdot 32 \cdot 3}{15 \cdot \delta \cdot 4} = \frac{8}{5}$ and $\bar{y} = \frac{M_x}{M} = \frac{4 \cdot \delta \cdot 3}{3 \cdot 4 \cdot \delta} = 1$. Therefore, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 1\right)$.

37. A typical horizontal strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{y^2+2y}{2}, y\right)$, length: $2y - y^2$, width: dy, area: $dA = (2y - y^2)\,dy$, mass: $dm = \delta \cdot dA = (1+y)(2y - y^2)\,dy \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = y(1+y)(2y-y^2)\,dy = (2y^2 + 2y^3 - y^3 - y^4)\,dy = (2y^2 + y^3 - y^4)\,dy$; the moment about the y-axis is



$\tilde{x}\,dm = \left(\frac{y^2+2y}{2}\right)(1+y)(2y-y^2)\,dy = \frac{1}{2}(4y^2 - y^4)(1+y)\,dy = \frac{1}{2}(4y^2 + 4y^3 - y^4 - y^5)\,dy$

$\Rightarrow M_x = \int \tilde{y}\,dm = \int_0^2 (2y^2 + y^3 - y^4)\,dy = \left[\frac{2}{3}y^3 + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^2 = \left(\frac{16}{3} + \frac{16}{4} - \frac{32}{5}\right) = 16\left(\frac{1}{3} + \frac{1}{4} - \frac{2}{5}\right)$

$= \frac{16}{60}(20 + 15 - 24) = \frac{4}{15}(11) = \frac{44}{15}$; $M_y = \int \tilde{x}\,dm = \int_0^2 \frac{1}{2}(4y^2 + 4y^3 - y^4 - y^5)\,dy = \frac{1}{2}\left[\frac{4}{3}y^3 + y^4 - \frac{y^5}{5} - \frac{y^6}{6}\right]_0^2$

$= \frac{1}{2}\left(\frac{4\cdot 2^3}{3} + 2^4 - \frac{2^5}{5} - \frac{2^6}{6}\right) = 4\left(\frac{4}{3} + 2 - \frac{4}{5} - \frac{8}{6}\right) = 4\left(2 - \frac{4}{5}\right) = \frac{24}{5}$; $M = \int dm = \int_0^2 (1+y)(2y-y^2)\,dy$

$= \int_0^2 (2y + y^2 - y^3)\,dy = \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2 = \left(4 + \frac{8}{3} - \frac{16}{4}\right) = \frac{8}{3} \Rightarrow \bar{x} = \frac{M_y}{M} = \left(\frac{24}{5}\right)\left(\frac{3}{8}\right) = \frac{9}{5}$ and $\bar{y} = \frac{M_x}{M}$

$= \left(\frac{44}{15}\right)\left(\frac{3}{8}\right) = \frac{44}{40} = \frac{11}{10}$. Therefore, the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{9}{5}, \frac{11}{10}\right)$.

38. A typical vertical strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{3}{2x^{3/2}}\right)$, length: $\frac{3}{x^{3/2}}$, width: dx, area: $dA = \frac{3}{x^{3/2}}\,dx$, mass: $dm = \delta \cdot dA = \delta \cdot \frac{3}{x^{3/2}}\,dx \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \frac{3}{2x^{3/2}} \cdot \delta \frac{3}{x^{3/2}}\,dx = \frac{9\delta}{2x^3}\,dx$; the moment about the y-axis is $\tilde{x}\,dm = x \cdot \delta \frac{3}{x^{3/2}}\,dx = \frac{3\delta}{x^{1/2}}\,dx$.

(a) $M_x = \delta \int_1^9 \frac{1}{2}\left(\frac{9}{x^3}\right)dx = \frac{9\delta}{2}\left[-\frac{x^{-2}}{2}\right]_1^9 = \frac{20\delta}{9}$; $M_y = \delta \int_1^9 x\left(\frac{3}{x^{3/2}}\right)dx = 3\delta\left[2x^{1/2}\right]_1^9 = 12\delta$;

$M = \delta \int_1^9 \frac{3}{x^{3/2}}\,dx = -6\delta\left[x^{-1/2}\right]_1^9 = 4\delta \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{12\delta}{4\delta} = 3$ and $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{20\delta}{9}\right)}{4\delta} = \frac{5}{9}$

(b) $M_x = \int_1^9 \frac{x}{2}\left(\frac{9}{x^3}\right)dx = \frac{9}{2}\left[-\frac{1}{x}\right]_1^9 = 4$; $M_y = \int_1^9 x^2\left(\frac{3}{x^{3/2}}\right)dx = \left[2x^{3/2}\right]_1^9 = 52$; $M = \int_1^9 x\left(\frac{3}{x^{3/2}}\right)dx$

$= 6\left[x^{1/2}\right]_1^9 = 12 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{13}{3}$ and $\bar{y} = \frac{M_x}{M} = \frac{1}{3}$

39. $F = \int_a^b W \cdot \binom{\text{strip}}{\text{depth}} \cdot L(y)\,dy \Rightarrow F = 2\int_0^2 (62.4)(2-y)(2y)\,dy = 249.6\int_0^2 (2y - y^2)\,dy = 249.6\left[y^2 - \frac{y^3}{3}\right]_0^2$

$= (249.6)\left(4 - \frac{8}{3}\right) = (249.6)\left(\frac{4}{3}\right) = 332.8$ lb

40. $F = \int_a^b W \cdot \binom{\text{strip}}{\text{depth}} \cdot L(y)\,dy \Rightarrow F = \int_0^{5/6} 75\left(\frac{5}{6} - y\right)(2y+4)\,dy = 75\int_0^{5/6}\left(\frac{5}{3}y + \frac{10}{3} - 2y^2 - 4y\right)dy$

$= 75\int_0^{5/6}\left(\frac{10}{3} - \frac{7}{3}y - 2y^2\right)dy = 75\left[\frac{10}{3}y - \frac{7}{6}y^2 - \frac{2}{3}y^3\right]_0^{5/6} = (75)\left[\left(\frac{50}{18}\right) - \left(\frac{7}{6}\right)\left(\frac{25}{36}\right) - \left(\frac{2}{3}\right)\left(\frac{125}{216}\right)\right]$

$= (75)\left(\frac{25}{9} - \frac{175}{216} - \frac{250}{3\cdot 216}\right) = \left(\frac{75}{9\cdot 216}\right)(25\cdot 216 - 175\cdot 9 - 250\cdot 3) = \frac{(75)(3075)}{9\cdot 216} \approx 118.63$ lb.

41. $F = \int_a^b W \cdot \binom{\text{strip}}{\text{depth}} \cdot L(y)\,dy \Rightarrow F = 62.4\int_0^4 (9-y)\left(2 \cdot \frac{\sqrt{y}}{2}\right)dy = 62.4\int_0^4 \left(9y^{1/2} - 3y^{3/2}\right)dy$

$= 62.4\left[6y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4 = (62.4)\left(6\cdot 8 - \frac{2}{5}\cdot 32\right) = \left(\frac{62.4}{5}\right)(48\cdot 5 - 64) = \frac{(62.4)(176)}{5} = 2196.48$ lb

42. Place the origin at the bottom of the tank. Then $F = \int_0^h W \cdot \binom{\text{strip}}{\text{depth}} \cdot L(y)\,dy$, h = the height of the mercury column, strip depth $= h - y$, $L(y) = 1 \Rightarrow F = \int_0^h 849(h-y)\,1\,dy = (849)\int_0^h (h-y)\,dy = 849\left[hy - \frac{y^2}{2}\right]_0^h = 849\left(h^2 - \frac{h^2}{2}\right)$

$= \frac{849}{2}h^2$. Now solve $\frac{849}{2}h^2 = 40000$ to get $h \approx 9.707$ ft. The volume of the mercury is $s^2h = 1^2 \cdot 9.707 = 9.707\ \text{ft}^3$.

CHAPTER 6 ADDITIONAL AND ADVANCED EXERCISES

1. $V = \pi \int_a^b [f(x)]^2\, dx = b^2 - ab \ \Rightarrow\ \pi \int_a^x [f(t)]^2\, dt = x^2 - ax$ for all $x > a \ \Rightarrow\ \pi [f(x)]^2 = 2x - a \ \Rightarrow\ f(x) = \sqrt{\frac{2x-a}{\pi}}$

2. $V = \pi \int_0^a [f(x)]^2\, dx = a^2 + a \ \Rightarrow\ \pi \int_0^x [f(t)]^2\, dt = x^2 + x$ for all $x > a \ \Rightarrow\ \pi[f(x)]^2 = 2x + 1 \ \Rightarrow\ f(x) = \sqrt{\frac{2x+1}{\pi}}$

3. $s(x) = Cx \Rightarrow \int_0^x \sqrt{1 + [f'(t)]^2}\, dt = Cx \Rightarrow \sqrt{1 + [f'(x)]^2} = C \Rightarrow f'(x) = \sqrt{C^2 - 1}$ for $C \ge 1$
$\Rightarrow f(x) = \int_0^x \sqrt{C^2 - 1}\, dt + k$. Then $f(0) = a \ \Rightarrow\ a = 0 + k \ \Rightarrow\ f(x) = \int_0^x \sqrt{C^2 - 1}\, dt + a \ \Rightarrow\ f(x) = x\sqrt{C^2 - 1} + a$, where $C \ge 1$.

4. (a) The graph of $f(x) = \sin x$ traces out a path from $(0, 0)$ to $(\alpha, \sin \alpha)$ whose length is $L = \int_0^\alpha \sqrt{1 + \cos^2 \theta}\, d\theta$. The line segment from $(0, 0)$ to $(\alpha, \sin \alpha)$ has length $\sqrt{(\alpha - 0)^2 + (\sin \alpha - 0)^2} = \sqrt{\alpha^2 + \sin^2 \alpha}$. Since the shortest distance between two points is the length of the straight line segment joining them, we have immediately that $\int_0^\alpha \sqrt{1 + \cos^2 \theta}\, d\theta > \sqrt{\alpha^2 + \sin^2 \alpha}$ if $0 < \alpha \le \frac{\pi}{2}$.
(b) In general, if $y = f(x)$ is continuously differentiable and $f(0) = 0$, then $\int_0^\alpha \sqrt{1 + [f'(t)]^2}\, dt > \sqrt{\alpha^2 + f^2(\alpha)}$ for $\alpha > 0$.

5. We can find the centroid and then use Pappus' Theorem to calculate the volume. $f(x) = x$, $g(x) = x^2$, $f(x) = g(x)$
$\Rightarrow x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x = 0, x = 1; \delta = 1; M = \int_0^1 [x - x^2]dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - 0 = \frac{1}{6}$
$\overline{x} = \frac{1}{1/6}\int_0^1 x[x - x^2]dx = 6\int_0^1 [x^2 - x^3]dx = 6\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) - 0 = \frac{1}{2}$
$\overline{y} = \frac{1}{1/6}\int_0^1 \frac{1}{2}\left[x^2 - (x^2)^2\right]dx = 3\int_0^1 [x^2 - x^4]dx = 3\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = 3\left(\frac{1}{3} - \frac{1}{5}\right) - 0 = \frac{2}{5} \Rightarrow$ The centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$.
ρ is the distance from $\left(\frac{1}{2}, \frac{2}{5}\right)$ to the axis of rotation, $y = x$. To calculate this distance we must find the point on $y = x$ that also lies on the line perpendicular to $y = x$ that passes through $\left(\frac{1}{2}, \frac{2}{5}\right)$. The equation of this line is $y - \frac{2}{5} = -1\left(x - \frac{1}{2}\right)$
$\Rightarrow x + y = \frac{9}{10}$. The point of intersection of the lines $x + y = \frac{9}{10}$ and $y = x$ is $\left(\frac{9}{20}, \frac{9}{20}\right)$. Thus,
$\rho = \sqrt{\left(\frac{9}{20} - \frac{1}{2}\right)^2 + \left(\frac{9}{20} - \frac{2}{5}\right)^2} = \frac{1}{10\sqrt{2}}$. Thus $V = 2\pi\left(\frac{1}{10\sqrt{2}}\right)\left(\frac{1}{6}\right) = \frac{\pi}{30\sqrt{2}}$.

6. Since the slice is made at an angle of $45°$, the volume of the wedge is half the volume of the cylinder of radius $\frac{1}{2}$ and height 1. Thus, $V = \frac{1}{2}\left[\pi\left(\frac{1}{2}\right)^2(1)\right] = \frac{\pi}{8}$.

7. $y = 2\sqrt{x} \ \Rightarrow\ ds = \sqrt{\frac{1}{x} + 1}\, dx \ \Rightarrow\ A = \int_0^3 2\sqrt{x}\sqrt{\frac{1}{x} + 1}\, dx = \frac{4}{3}\left[(1 + x)^{3/2}\right]_0^3 = \frac{28}{3}$

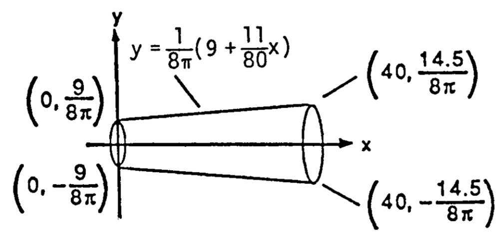
8. This surface is a triangle having a base of $2\pi a$ and a height of $2\pi ak$. Therefore the surface area is $\frac{1}{2}(2\pi a)(2\pi ak) = 2\pi^2 a^2 k$.

9. $F = ma = t^2 \ \Rightarrow\ \frac{d^2x}{dt^2} = a = \frac{t^2}{m} \ \Rightarrow\ v = \frac{dx}{dt} = \frac{t^3}{3m} + C$; $v = 0$ when $t = 0 \ \Rightarrow\ C = 0 \ \Rightarrow\ \frac{dx}{dt} = \frac{t^3}{3m} \ \Rightarrow\ x = \frac{t^4}{12m} + C_1$;
$x = 0$ when $t = 0 \ \Rightarrow\ C_1 = 0 \ \Rightarrow\ x = \frac{t^4}{12m}$. Then $x = h \ \Rightarrow\ t = (12mh)^{1/4}$. The work done is
$W = \int F\, dx = \int_0^{(12mh)^{1/4}} F(t) \cdot \frac{dx}{dt}\, dt = \int_0^{(12mh)^{1/4}} t^2 \cdot \frac{t^3}{3m}\, dt = \frac{1}{3m}\left[\frac{t^6}{6}\right]_0^{(12mh)^{1/4}} = \left(\frac{1}{18m}\right)(12mh)^{6/4}$
$= \frac{(12mh)^{3/2}}{18m} = \frac{12mh \cdot \sqrt{12mh}}{18m} = \frac{2h}{3} \cdot 2\sqrt{3mh} = \frac{4h}{3}\sqrt{3mh}$

10. Converting to pounds and feet, 2 lb/in $= \frac{2 \text{ lb}}{1 \text{ in}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 24$ lb/ft. Thus, $F = 24x \Rightarrow W = \int_0^{1/2} 24x\,dx$ $= \left[12x^2\right]_0^{1/2} = 3$ ft · lb. Since $W = \frac{1}{2} mv_0^2 - \frac{1}{2} mv_1^2$, where $W = 3$ ft · lb, $m = \left(\frac{1}{10} \text{ lb}\right)\left(\frac{1}{32 \text{ ft/sec}^2}\right)$ $= \frac{1}{320}$ slugs, and $v_1 = 0$ ft/sec, we have $3 = \left(\frac{1}{2}\right)\left(\frac{1}{320} v_0^2\right) \Rightarrow v_0^2 = 3 \cdot 640$. For the projectile height, $s = -16t^2 + v_0 t$ (since $s = 0$ at $t = 0$) $\Rightarrow \frac{ds}{dt} = v = -32t + v_0$. At the top of the ball's path, $v = 0 \Rightarrow t = \frac{v_0}{32}$ and the height is $s = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = \frac{v_0^2}{64} = \frac{3 \cdot 640}{64} = 30$ ft.

11. From the symmetry of $y = 1 - x^n$, n even, about the y-axis for $-1 \le x \le 1$, we have $\overline{x} = 0$. To find $\overline{y} = \frac{M_x}{M}$, we use the vertical strips technique. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{1-x^n}{2}\right)$, length: $1 - x^n$, width: dx, area: $dA = (1 - x^n)\,dx$, mass: $dm = 1 \cdot dA = (1 - x^n)\,dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm = \frac{(1-x^n)^2}{2}\,dx \Rightarrow M_x = \int_{-1}^{1} \frac{(1-x^n)^2}{2}\,dx = 2\int_0^1 \frac{1}{2}\left(1 - 2x^n + x^{2n}\right)dx = \left[x - \frac{2x^{n+1}}{n+1} + \frac{x^{2n+1}}{2n+1}\right]_0^1$ $= 1 - \frac{2}{n+1} + \frac{1}{2n+1} = \frac{(n+1)(2n+1) - 2(2n+1) + (n+1)}{(n+1)(2n+1)} = \frac{2n^2 + 3n + 1 - 4n - 2 + n + 1}{(n+1)(2n+1)} = \frac{2n^2}{(n+1)(2n+1)}$.
Also, $M = \int_{-1}^{1} dA = \int_{-1}^{1} (1 - x^n)\,dx = 2\int_0^1 (1 - x^n)\,dx = 2\left[x - \frac{x^{n+1}}{n+1}\right]_0^1 = 2\left(1 - \frac{1}{n+1}\right) = \frac{2n}{n+1}$. Therefore, $\overline{y} = \frac{M_x}{M} = \frac{2n^2}{(n+1)(2n+1)} \cdot \frac{(n+1)}{2n} = \frac{n}{2n+1} \Rightarrow \left(0, \frac{n}{2n+1}\right)$ is the location of the centroid. As $n \to \infty$, $\overline{y} \to \frac{1}{2}$ so the limiting position of the centroid is $\left(0, \frac{1}{2}\right)$.

12. Align the telephone pole along the x-axis as shown in the accompanying figure. The slope of the top length of pole is $\frac{\left(\frac{14.5}{8\pi} - \frac{9}{8\pi}\right)}{40} = \frac{1}{8\pi} \cdot \frac{1}{40} \cdot (14.5 - 9) = \frac{5.5}{8\pi \cdot 40} = \frac{11}{8\pi \cdot 80}$. Thus, $y = \frac{9}{8\pi} + \frac{11}{8\pi \cdot 80}x = \frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)$ is an equation of the line representing the top of the pole. Then,



$M_y = \int_a^b x \cdot \pi y^2\,dx = \pi \int_0^{40} x\left[\frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)\right]^2 dx$ $= \frac{1}{64\pi}\int_0^{40} x\left(9 + \frac{11}{80}x\right)^2 dx$; $M = \int_a^b \pi y^2\,dx$ $= \pi \int_0^{40}\left[\frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)\right]^2 dx = \frac{1}{64\pi}\int_0^{40}\left(9 + \frac{11}{80}x\right)^2 dx$. Thus, $\overline{x} = \frac{M_y}{M} \approx \frac{129{,}700}{5623.3} \approx 23.06$ (using a calculator to compute the integrals). By symmetry about the x-axis, $\overline{y} = 0$ so the center of mass is about 23 ft from the top of the pole.

13. (a) Consider a single vertical strip with center of mass $(\tilde{x}, \tilde{y})$. If the plate lies to the right of the line, then the moment of this strip about the line $x = b$ is $(\tilde{x} - b)\,dm = (\tilde{x} - b)\,\delta\,dA \Rightarrow$ the plate's first moment about $x = b$ is the integral $\int (x - b)\delta\,dA = \int \delta x\,dA - \int \delta b\,dA = M_y - b\delta A$.
(b) If the plate lies to the left of the line, the moment of a vertical strip about the line $x = b$ is $(b - \tilde{x})\,dm = (b - \tilde{x})\,\delta\,dA \Rightarrow$ the plate's first moment about $x = b$ is $\int (b - x)\delta\,dA = \int b\delta\,dA - \int \delta x\,dA$ $= b\delta A - M_y$.

14. (a) By symmetry of the plate about the x-axis, $\overline{y} = 0$. A typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = (x, 0)$, length: $4\sqrt{ax}$, width: dx, area: $4\sqrt{ax}\,dx$, mass: $dm = \delta\,dA = kx \cdot 4\sqrt{ax}\,dx$, for some proportionality constant k. The moment of the strip about the y-axis is $M_y = \int \tilde{x}\,dm = \int_0^a 4kx^2\sqrt{ax}\,dx$ $= 4k\sqrt{a}\int_0^a x^{5/2}\,dx = 4k\sqrt{a}\left[\frac{2}{7}x^{7/2}\right]_0^a = 4ka^{1/2} \cdot \frac{2}{7}a^{7/2} = \frac{8ka^4}{7}$. Also, $M = \int dm = \int_0^a 4kx\sqrt{ax}\,dx$ $= 4k\sqrt{a}\int_0^a x^{3/2}\,dx = 4k\sqrt{a}\left[\frac{2}{5}x^{5/2}\right]_0^a = 4ka^{1/2} \cdot \frac{2}{5}a^{5/2} = \frac{8ka^3}{5}$. Thus, $\overline{x} = \frac{M_y}{M} = \frac{8ka^4}{7} \cdot \frac{5}{8ka^3} = \frac{5}{7}a$ $\Rightarrow (\overline{x}, \overline{y}) = \left(\frac{5a}{7}, 0\right)$ is the center of mass.
(b) A typical horizontal strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{\frac{y^2}{4a} + a}{2}, y\right) = \left(\frac{y^2 + 4a^2}{8a}, y\right)$, length: $a - \frac{y^2}{4a}$, width: dy, area: $\left(a - \frac{y^2}{4a}\right)dy$, mass: $dm = \delta\,dA = |y|\left(a - \frac{y^2}{4a}\right)dy$. Thus, $M_x = \int \tilde{y}\,dm$ $= \int_{-2a}^{2a} y\,|y|\left(a - \frac{y^2}{4a}\right)dy = \int_{-2a}^{0} -y^2\left(a - \frac{y^2}{4a}\right)dy + \int_0^{2a} y^2\left(a - \frac{y^2}{4a}\right)dy$

$= \int_{-2a}^{0} \left(-ay^2 + \frac{y^4}{4a}\right) dy + \int_{0}^{2a} \left(ay^2 - \frac{y^4}{4a}\right) dy = \left[-\frac{a}{3}y^3 + \frac{y^5}{20a}\right]_{-2a}^{0} + \left[\frac{a}{3}y^3 - \frac{y^5}{20a}\right]_{0}^{2a}$

$= -\frac{8a^4}{3} + \frac{32a^5}{20a} + \frac{8a^4}{3} - \frac{32a^5}{20a} = 0$; $M_y = \int \widetilde{x}\, dm = \int_{-2a}^{2a} \left(\frac{y^2+4a^2}{8a}\right) |y| \left(a - \frac{y^2}{4a}\right) dy$

$= \frac{1}{8a} \int_{-2a}^{2a} |y| \left(y^2 + 4a^2\right) \left(\frac{4a^2 - y^2}{4a}\right) dy = \frac{1}{32a^2} \int_{-2a}^{2a} |y| \left(16a^4 - y^4\right) dy$

$= \frac{1}{32a^2} \int_{-2a}^{0} \left(-16a^4 y + y^5\right) dy + \frac{1}{32a^2} \int_{0}^{2a} \left(16a^4 y - y^5\right) dy = \frac{1}{32a^2} \left[-8a^4y^2 + \frac{y^6}{6}\right]_{-2a}^{0} + \frac{1}{32a^2} \left[8a^4y^2 - \frac{y^6}{6}\right]_{0}^{2a}$

$= \frac{1}{32a^2} \left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] + \frac{1}{32a^2} \left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] = \frac{1}{16a^2} \left(32a^6 - \frac{32a^6}{3}\right) = \frac{1}{16a^2} \cdot \frac{2}{3} \left(32a^6\right) = \frac{4}{3} a^4$;

$M = \int dm = \int_{-2a}^{2a} |y| \left(\frac{4a^2 - y^2}{4a}\right) dy = \frac{1}{4a} \int_{-2a}^{2a} |y| \left(4a^2 - y^2\right) dy$

$= \frac{1}{4a} \int_{-2a}^{0} \left(-4a^2 y + y^3\right) dy + \frac{1}{4a} \int_{0}^{2a} \left(4a^2 y - y^3\right) dy = \frac{1}{4a} \left[-2a^2y^2 + \frac{y^4}{4}\right]_{-2a}^{0} + \frac{1}{4a} \left[2a^2y^2 - \frac{y^4}{4}\right]_{0}^{2a}$

$= 2 \cdot \frac{1}{4a} \left(2a^2 \cdot 4a^2 - \frac{16a^4}{4}\right) = \frac{1}{2a} \left(8a^4 - 4a^4\right) = 2a^3$. Therefore, $\overline{x} = \frac{M_y}{M} = \left(\frac{4}{3}a^4\right)\left(\frac{1}{2a^3}\right) = \frac{2a}{3}$ and

$\overline{y} = \frac{M_x}{M} = 0$ is the center of mass.

15. (a) On $[0, a]$ a typical *vertical* strip has center of mass: $\left(\widetilde{x}, \widetilde{y}\right) = \left(x, \frac{\sqrt{b^2 - x^2} + \sqrt{a^2 - x^2}}{2}\right)$,

length: $\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}$, width: dx, area: $dA = \left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx$, mass: $dm = \delta\, dA$

$= \delta \left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx$. On $[a, b]$ a typical *vertical* strip has center of mass:

$\left(\widetilde{x}, \widetilde{y}\right) = \left(x, \frac{\sqrt{b^2 - x^2}}{2}\right)$, length: $\sqrt{b^2 - x^2}$, width: dx, area: $dA = \sqrt{b^2 - x^2}\, dx$,

mass: $dm = \delta\, dA = \delta \sqrt{b^2 - x^2}\, dx$. Thus, $M_x = \int \widetilde{y}\, dm$

$= \int_{0}^{a} \frac{1}{2} \left(\sqrt{b^2 - x^2} + \sqrt{a^2 - x^2}\right) \delta \left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx + \int_{a}^{b} \frac{1}{2} \sqrt{b^2 - x^2}\, \delta \sqrt{b^2 - x^2}\, dx$

$= \frac{\delta}{2} \int_{0}^{a} \left[\left(b^2 - x^2\right) - \left(a^2 - x^2\right)\right] dx + \frac{\delta}{2} \int_{a}^{b} \left(b^2 - x^2\right) dx = \frac{\delta}{2} \int_{0}^{a} \left(b^2 - a^2\right) dx + \frac{\delta}{2} \int_{a}^{b} \left(b^2 - x^2\right) dx$

$= \frac{\delta}{2} \left[\left(b^2 - a^2\right) x\right]_{0}^{a} + \frac{\delta}{2} \left[b^2 x - \frac{x^3}{3}\right]_{a}^{b} = \frac{\delta}{2} \left[\left(b^2 - a^2\right) a\right] + \frac{\delta}{2} \left[\left(b^3 - \frac{b^3}{3}\right) - \left(b^2 a - \frac{a^3}{3}\right)\right]$

$= \frac{\delta}{2} \left(ab^2 - a^3\right) + \frac{\delta}{2} \left(\frac{2}{3} b^3 - ab^2 + \frac{a^3}{3}\right) = \frac{\delta b^3}{3} - \frac{\delta a^3}{3} = \delta \left(\frac{b^3 - a^3}{3}\right)$; $M_y = \int \widetilde{x}\, dm$

$= \int_{0}^{a} x\delta \left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx + \int_{a}^{b} x\delta \sqrt{b^2 - x^2}\, dx$

$= \delta \int_{0}^{a} x \left(b^2 - x^2\right)^{1/2} dx - \delta \int_{0}^{a} x \left(a^2 - x^2\right)^{1/2} dx + \delta \int_{a}^{b} x \left(b^2 - x^2\right)^{1/2} dx$

$= \frac{-\delta}{2} \left[\frac{2\left(b^2 - x^2\right)^{3/2}}{3}\right]_{0}^{a} + \frac{\delta}{2} \left[\frac{2\left(a^2 - x^2\right)^{3/2}}{3}\right]_{0}^{a} - \frac{\delta}{2} \left[\frac{2\left(b^2 - x^2\right)^{3/2}}{3}\right]_{a}^{b}$

$= -\frac{\delta}{3} \left[\left(b^2 - a^2\right)^{3/2} - \left(b^2\right)^{3/2}\right] + \frac{\delta}{3} \left[0 - \left(a^2\right)^{3/2}\right] - \frac{\delta}{3} \left[0 - \left(b^2 - a^2\right)^{3/2}\right] = \frac{\delta b^3}{3} - \frac{\delta a^3}{3} = \frac{\delta\left(b^3 - a^3\right)}{3} = M_x$;

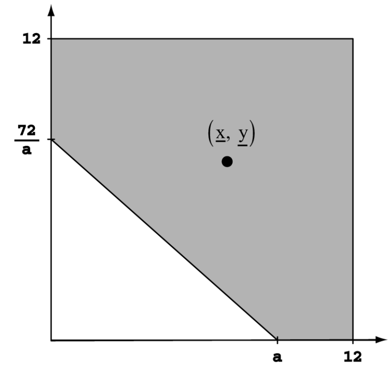
We calculate the mass geometrically: $M = \delta A = \delta \left(\frac{\pi b^2}{4}\right) - \delta \left(\frac{\pi a^2}{4}\right) = \frac{\delta \pi}{4} \left(b^2 - a^2\right)$. Thus, $\overline{x} = \frac{M_y}{M}$

$= \frac{\delta\left(b^3 - a^3\right)}{3} \cdot \frac{4}{\delta\pi\left(b^2 - a^2\right)} = \frac{4}{3\pi} \left(\frac{b^3 - a^3}{b^2 - a^2}\right) = \frac{4}{3\pi} \frac{(b - a)\left(a^2 + ab + b^2\right)}{(b - a)(b + a)} = \frac{4\left(a^2 + ab + b^2\right)}{3\pi(a + b)}$; likewise

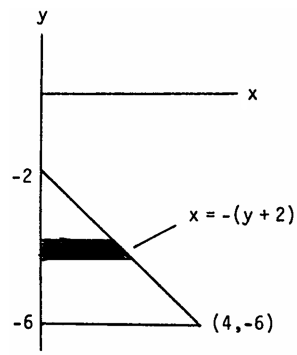
$\overline{y} = \frac{M_x}{M} = \frac{4\left(a^2 + ab + b^2\right)}{3\pi(a + b)}$.

(b) $\lim_{b \to a} \frac{4}{3\pi} \left(\frac{a^2 + ab + b^2}{a + b}\right) = \left(\frac{4}{3\pi}\right) \left(\frac{a^2 + a^2 + a^2}{a + a}\right) = \left(\frac{4}{3\pi}\right) \left(\frac{3a^2}{2a}\right) = \frac{2a}{\pi} \Rightarrow \left(\overline{x}, \overline{y}\right) = \left(\frac{2a}{\pi}, \frac{2a}{\pi}\right)$ is the limiting position of the centroid as $b \to a$. This is the centroid of a circle of radius a (and we note the two circles coincide when b = a).

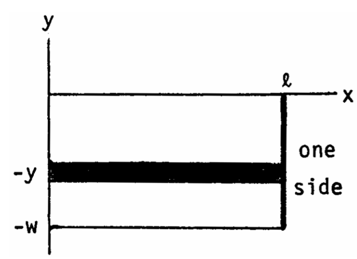
16. Since the area of the traingle is 36, the diagram may be labeled as shown at the right. The centroid of the triangle is $\left(\frac{a}{3}, \frac{24}{a}\right)$. The shaded portion is $144 - 36 = 108$. Write $(\underline{x}, \underline{y})$ for the centroid of the remaining region. The centroid of the whole square is obviously $(6, 6)$. Think of the square as a sheet of uniform density, so that the centroid of the square is the average of the centroids of the two regions, weighted by area:
$6 = \frac{36\left(\frac{a}{3}\right) + 108(\underline{x})}{144}$ and $6 = \frac{36\left(\frac{24}{a}\right) + 108(\underline{y})}{144}$
which we solve to get $\underline{x} = 8 - \frac{a}{9}$ and $\underline{y} = \frac{8(a-1)}{a}$. Set $\underline{x} = 7$ in. (Given). It follows that $a = 9$, whence $\underline{y} = \frac{64}{9}$ $= 7\frac{1}{9}$ in. The distances of the centroid $(\underline{x}, \underline{y})$ from the other sides are easily computed. (Note that if we set $\underline{y} = 7$ in. above, we will find $\underline{x} = 7\frac{1}{9}$.)



17. The submerged triangular plate is depicted in the figure at the right. The hypotenuse of the triangle has slope -1 $\Rightarrow\ y - (-2) = -(x - 0)\ \Rightarrow\ x = -(y + 2)$ is an equation of the hypotenuse. Using a typical horizontal strip, the fluid pressure is $F = \int (62.4) \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{length}\end{smallmatrix}\right) dy$
$= \int_{-6}^{-2} (62.4)(-y)[-(y+2)]\, dy = 62.4 \int_{-6}^{-2} (y^2 + 2y)\, dy$
$= 62.4 \left[\frac{y^3}{3} + y^2\right]_{-6}^{-2} = (62.4)\left[\left(-\frac{8}{3} + 4\right) - \left(-\frac{216}{3} + 36\right)\right]$
$= (62.4)\left(\frac{208}{3} - 32\right) = \frac{(62.4)(112)}{3} \approx 2329.6$ lb



18. Consider a rectangular plate of length ℓ and width w. The length is parallel with the surface of the fluid of weight density ω. The force on one side of the plate is $F = \omega \int_{-w}^{0} (-y)(\ell)\, dy = -\omega\ell \left[\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega \ell w^2}{2}$. The average force on one side of the plate is $F_{av} = \frac{\omega}{w} \int_{-w}^{0} (-y)dy$ $= \frac{\omega}{w}\left[-\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega w}{2}$. Therefore the force $\frac{\omega \ell w^2}{2}$ $= \left(\frac{\omega w}{2}\right)(\ell w) =$ (the average pressure up and down) $\cdot$ (the area of the plate).



NOTES: