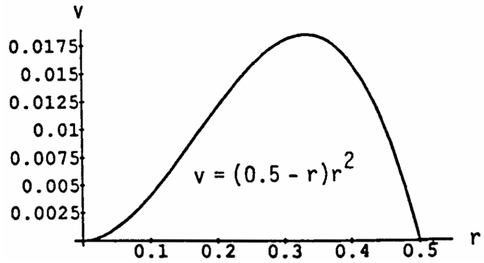
60. (a) If $v = cr_0r^2 - cr^3$, then $v' = 2cr_0r - 3cr^2 = cr(2r_0 - 3r)$ and $v'' = 2cr_0 - 6cr = 2c(r_0 - 3r)$. The solution of $v' = 0$ is $r = 0$ or $\frac{2r_0}{3}$, but 0 is not in the domain. Also, $v' > 0$ for $r < \frac{2r_0}{3}$ and $v' < 0$ for $r > \frac{2r_0}{3} \Rightarrow$ at $r = \frac{2r_0}{3}$ there is a maximum.

(b) The graph confirms the findings in (a).



61. If $x > 0$, then $(x-1)^2 \geq 0 \Rightarrow x^2 + 1 \geq 2x \Rightarrow \frac{x^2+1}{x} \geq 2$. In particular if a, b, c and d are positive integers, then $\left(\frac{a^2+1}{a}\right)\left(\frac{b^2+1}{b}\right)\left(\frac{c^2+1}{c}\right)\left(\frac{d^2+1}{d}\right) \geq 16$.

62. (a) $f(x) = \frac{x}{\sqrt{a^2+x^2}} \Rightarrow f'(x) = \frac{(a^2+x^2)^{1/2} - x^2(a^2+x^2)^{-1/2}}{(a^2+x^2)} = \frac{a^2+x^2-x^2}{(a^2+x^2)^{3/2}} = \frac{a^2}{(a^2+x^2)^{3/2}} > 0$
$\Rightarrow$ f(x) is an increasing function of x

(b) $g(x) = \frac{d-x}{\sqrt{b^2+(d-x)^2}} \Rightarrow g'(x) = \frac{-(b^2+(d-x)^2)^{1/2} + (d-x)^2(b^2+(d-x)^2)^{-1/2}}{b^2+(d-x)^2}$
$= \frac{-(b^2+(d-x)^2)+(d-x)^2}{(b^2+(d-x)^2)^{3/2}} = \frac{-b^2}{(b^2+(d-x)^2)^{3/2}} < 0 \Rightarrow$ g(x) is a decreasing function of x

(c) Since $c_1, c_2 > 0$, the derivative $\frac{dt}{dx}$ is an increasing function of x (from part (a)) minus a decreasing function of x (from part (b)): $\frac{dt}{dx} = \frac{1}{c_1}f(x) - \frac{1}{c_2}g(x) \Rightarrow \frac{d^2t}{dx^2} = \frac{1}{c_1}f'(x) - \frac{1}{c_2}g'(x) > 0$ since $f'(x) > 0$ and $g'(x) < 0 \Rightarrow \frac{dt}{dx}$ is an increasing function of x.

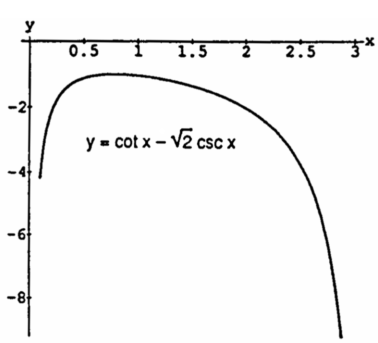
63. At $x = c$, the tangents to the curves are parallel. Justification: The vertical distance between the curves is $D(x) = f(x) - g(x)$, so $D'(x) = f'(x) - g'(x)$. The maximum value of D will occur at a point c where $D' = 0$. At such a point, $f'(c) - g'(c) = 0$, or $f'(c) = g'(c)$.

64. (a) $f(x) = 3 + 4\cos x + \cos 2x$ is a periodic function with period 2π

(b) No, $f(x) = 3 + 4\cos x + \cos 2x = 3 + 4\cos x + (2\cos^2 x - 1) = 2(1 + 2\cos x + \cos^2 x) = 2(1+\cos x)^2 \geq 0$
$\Rightarrow$ f(x) is never negative.

65. (a) If $y = \cot x - \sqrt{2}\csc x$ where $0 < x < \pi$, then $y' = (\csc x)\left(\sqrt{2}\cot x - \csc x\right)$. Solving $y' = 0 \Rightarrow \cos x = \frac{1}{\sqrt{2}}$ $\Rightarrow x = \frac{\pi}{4}$. For $0 < x < \frac{\pi}{4}$ we have $y' > 0$, and $y' < 0$ when $\frac{\pi}{4} < x < \pi$. Therefore, at $x = \frac{\pi}{4}$ there is a maximum value of $y = -1$.

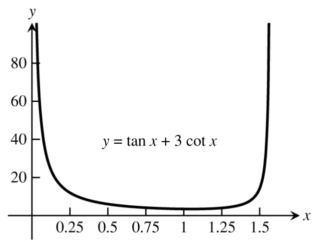
(b)



The graph confirms the findings in (a).

66. (a) If $y = \tan x + 3 \cot x$ where $0 < x < \frac{\pi}{2}$, then $y' = \sec^2 x - 3 \csc^2 x$. Solving $y' = 0 \Rightarrow \tan x = \pm\sqrt{3}$ $\Rightarrow x = \pm\frac{\pi}{3}$, but $-\frac{\pi}{3}$ is not in the domain. Also, $y'' = 2 \sec^2 x \tan x + 6 \csc^2 x \cot x > 0$ for all $0 < x < \frac{\pi}{2}$. Therefore at $x = \frac{\pi}{3}$ there is a minimum value of $y = 2\sqrt{3}$.

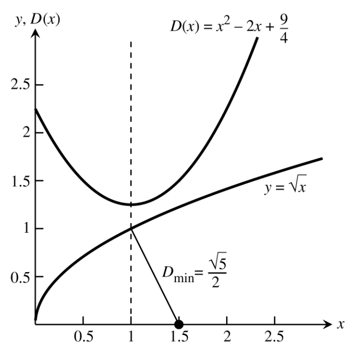
(b)



The graph confirms the findings in (a).

67. (a) The square of the distance is $D(x) = \left(x - \frac{3}{2}\right)^2 + \left(\sqrt{x} + 0\right)^2 = x^2 - 2x + \frac{9}{4}$, so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

(b)



The minimum distance is from the point $\left(\frac{3}{2}, 0\right)$ to the point $(1, 1)$ on the graph of $y = \sqrt{x}$, and this occurs at the value $x = 1$ where $D(x)$, the distance squared, has its minimum value.

68. (a) Calculus Method:

The square of the distance from the point $\left(1, \sqrt{3}\right)$ to $\left(x, \sqrt{16 - x^2}\right)$ is given by

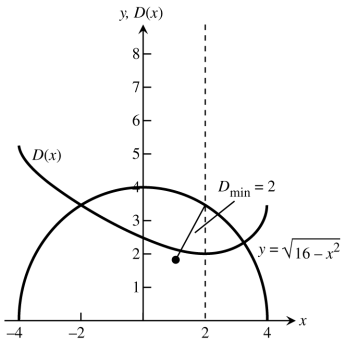
$D(x) = (x - 1)^2 + \left(\sqrt{16 - x^2} - \sqrt{3}\right)^2 = x^2 - 2x + 1 + 16 - x^2 - 2\sqrt{48 - 3x^2} + 3 = -2x + 20 - 2\sqrt{48 - 3x^2}$.

Then $D'(x) = -2 - \frac{1}{2} \cdot \frac{2}{\sqrt{48 - 3x^2}}(-6x) = -2 + \frac{6x}{\sqrt{48 - 3x^2}}$. Solving $D'(x) = 0$ we have: $6x = 2\sqrt{48 - 3x^2}$ $\Rightarrow 36x^2 = 4(48 - 3x^2) \Rightarrow 9x^2 = 48 - 3x^2 \Rightarrow 12x^2 = 48 \Rightarrow x = \pm 2$. We discard $x = -2$ as an extraneous solution, leaving $x = 2$. Since $D'(x) < 0$ for $-4 < x < 2$ and $D'(x) > 0$ for $2 < x < 4$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(2)} = 2$.

Geometry Method:

The semicircle is centered at the origin and has radius 4. The distance from the origin to $\left(1, \sqrt{3}\right)$ is $\sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$. The shortest distance from the point to the semicircle is the distance along the radius containing the point $\left(1, \sqrt{3}\right)$. That distance is $4 - 2 = 2$.

(b)



The minimum distance is from the point $\left(1, \sqrt{3}\right)$ to the point $\left(2, 2\sqrt{3}\right)$ on the graph of $y = \sqrt{16 - x^2}$, and this occurs at the value $x = 2$ where $D(x)$, the distance squared, has its minimum value.

4.6 NEWTON'S METHOD

1. $y = x^2 + x - 1 \Rightarrow y' = 2x + 1 \Rightarrow x_{n+1} = x_n - \frac{x_n^2 + x_n - 1}{2x_n + 1}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1+1-1}{2+1} = \frac{2}{3}$
$\Rightarrow x_2 = \frac{2}{3} - \frac{\frac{4}{9} + \frac{2}{3} - 1}{\frac{4}{3} + 1} \Rightarrow x_2 = \frac{2}{3} - \frac{4+6-9}{12+9} = \frac{2}{3} - \frac{1}{21} = \frac{13}{21} \approx .61905$; $x_0 = -1 \Rightarrow x_1 = 1 - \frac{1-1-1}{-2+1} = -2$
$\Rightarrow x_2 = -2 - \frac{4-2-1}{-4+1} = -\frac{5}{3} \approx -1.66667$

2. $y = x^3 + 3x + 1 \Rightarrow y' = 3x^2 + 3 \Rightarrow x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$; $x_0 = 0 \Rightarrow x_1 = 0 - \frac{1}{3} = -\frac{1}{3}$
$\Rightarrow x_2 = -\frac{1}{3} - \frac{-\frac{1}{27} - 1 + 1}{\frac{1}{3} + 3} = -\frac{1}{3} + \frac{1}{90} = -\frac{29}{90} \approx -0.32222$

3. $y = x^4 + x - 3 \Rightarrow y' = 4x^3 + 1 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1+1-3}{4+1} = \frac{6}{5}$
$\Rightarrow x_2 = \frac{6}{5} - \frac{\frac{1296}{625} + \frac{6}{5} - 3}{\frac{864}{125} + 1} = \frac{6}{5} - \frac{1296 + 750 - 1875}{4320 + 625} = \frac{6}{5} - \frac{171}{4945} = \frac{5763}{4945} \approx 1.16542$; $x_0 = -1 \Rightarrow x_1 = -1 - \frac{1-1-3}{-4+1}$
$= -2 \Rightarrow x_2 = -2 - \frac{16-2-3}{-32+1} = -2 + \frac{11}{31} = -\frac{51}{31} \approx -1.64516$

4. $y = 2x - x^2 + 1 \Rightarrow y' = 2 - 2x \Rightarrow x_{n+1} = x_n - \frac{2x_n - x_n^2 + 1}{2 - 2x_n}$; $x_0 = 0 \Rightarrow x_1 = 0 - \frac{0-0+1}{2-0} = -\frac{1}{2}$
$\Rightarrow x_2 = -\frac{1}{2} - \frac{-1 - \frac{1}{4} + 1}{2+1} = -\frac{1}{2} + \frac{1}{12} = -\frac{5}{12} \approx -.41667$; $x_0 = 2 \Rightarrow x_1 = 2 - \frac{4-4+1}{2-4} = \frac{5}{2} \Rightarrow x_2 = \frac{5}{2} - \frac{5 - \frac{25}{4} + 1}{2-5}$
$= \frac{5}{2} - \frac{20-25+4}{-12} = \frac{5}{2} - \frac{1}{12} = \frac{29}{12} \approx 2.41667$

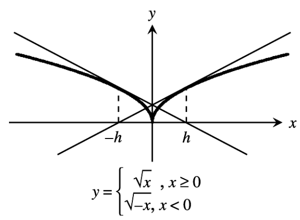
5. $y = x^4 - 2 \Rightarrow y' = 4x^3 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 - 2}{4x_n^3}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1-2}{4} = \frac{5}{4} \Rightarrow x_2 = \frac{5}{4} - \frac{\frac{625}{256} - 2}{\frac{125}{16}} = \frac{5}{4} - \frac{625-512}{2000}$
$= \frac{5}{4} - \frac{113}{2000} = \frac{2500-113}{2000} = \frac{2387}{2000} \approx 1.1935$

6. From Exercise 5, $x_{n+1} = x_n - \frac{x_n^4 - 2}{4x_n^3}$; $x_0 = -1 \Rightarrow x_1 = -1 - \frac{1-2}{-4} = -1 - \frac{1}{4} = -\frac{5}{4} \Rightarrow x_2 = -\frac{5}{4} - \frac{\frac{625}{256} - 2}{-\frac{125}{16}}$
$= -\frac{5}{4} - \frac{625-512}{-2000} = -\frac{5}{4} + \frac{113}{2000} \approx -1.1935$

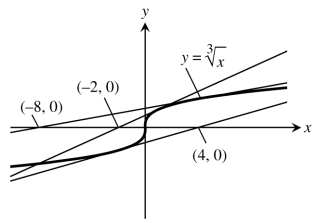
7. $f(x_0) = 0$ and $f'(x_0) \neq 0 \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ gives $x_1 = x_0 \Rightarrow x_2 = x_0 \Rightarrow x_n = x_0$ for all $n \geq 0$. That is, all of the approximations in Newton's method will be the root of $f(x) = 0$.

8. It does matter. If you start too far away from $x = \frac{\pi}{2}$, the calculated values may approach some other root. Starting with $x_0 = -0.5$, for instance, leads to $x = -\frac{\pi}{2}$ as the root, not $x = \frac{\pi}{2}$.

9. If $x_0 = h > 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = h - \frac{f(h)}{f'(h)}$
$= h - \frac{\sqrt{h}}{\left(\frac{1}{2\sqrt{h}}\right)} = h - \left(\sqrt{h}\right)\left(2\sqrt{h}\right) = -h;$
if $x_0 = -h < 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -h - \frac{f(-h)}{f'(-h)}$
$= -h - \frac{\sqrt{h}}{\left(\frac{-1}{2\sqrt{h}}\right)} = -h + \left(\sqrt{h}\right)\left(2\sqrt{h}\right) = h.$



10. $f(x) = x^{1/3} \Rightarrow f'(x) = \left(\frac{1}{3}\right)x^{-2/3} \Rightarrow x_{n+1} = x_n - \frac{x_n^{1/3}}{\left(\frac{1}{3}\right)x_n^{-2/3}}$
$= -2x_n$; $x_0 = 1 \Rightarrow x_1 = -2$, $x_2 = 4$, $x_3 = -8$, and
$x_4 = 16$ and so forth. Since $|x_n| = 2|x_{n-1}|$ we may conclude
that $n \to \infty \Rightarrow |x_n| \to \infty$.



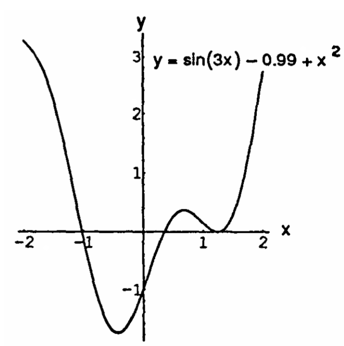
11. i) is equivalent to solving $x^3 - 3x - 1 = 0$.
ii) is equivalent to solving $x^3 - 3x - 1 = 0$.
iii) is equivalent to solving $x^3 - 3x - 1 = 0$.
iv) is equivalent to solving $x^3 - 3x - 1 = 0$.
All four equations are equivalent.

12. $f(x) = x - 1 - 0.5 \sin x \Rightarrow f'(x) = 1 - 0.5 \cos x \Rightarrow x_{n+1} = x_n - \frac{x_n - 1 - 0.5 \sin x_n}{1 - 0.5 \cos x_n}$; if $x_0 = 1.5$, then $x_1 = 1.49870$

13. $f(x) = \tan x - 2x \Rightarrow f'(x) = \sec^2 x - 2 \Rightarrow x_{n+1} = x_n - \frac{\tan(x_n) - 2x_n}{\sec^2(x_n)}$; $x_0 = 1 \Rightarrow x_1 = 1.2920445$
$\Rightarrow x_2 = 1.155327774 \Rightarrow x_{16} = x_{17} = 1.165561185$

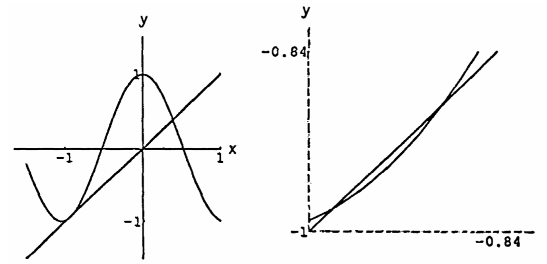
14. $f(x) = x^4 - 2x^3 - x^2 - 2x + 2 \Rightarrow f'(x) = 4x^3 - 6x^2 - 2x - 2 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 - 2x_n^3 - x_n^2 - 2x_n + 2}{4x_n^3 - 6x_n^2 - 2x_n - 2}$;
if $x_0 = 0.5$, then $x_4 = 0.630115396$; if $x_0 = 2.5$, then $x_4 = 2.57327196$

15. (a) The graph of $f(x) = \sin 3x - 0.99 + x^2$ in the window $-2 \le x \le 2$, $-2 \le y \le 3$ suggests three roots. However, when you zoom in on the x-axis near $x = 1.2$, you can see that the graph lies above the axis there. There are only two roots, one near $x = -1$, the other near $x = 0.4$.
(b) $f(x) = \sin 3x - 0.99 + x^2 \Rightarrow f'(x) = 3 \cos 3x + 2x$
$\Rightarrow x_{n+1} = x_n - \frac{\sin(3x_n) - 0.99 + x_n^2}{3\cos(3x_n) + 2x_n}$ and the solutions are approximately 0.35003501505249 and −1.0261731615301



16. (a) Yes, three times as indicted by the graphs

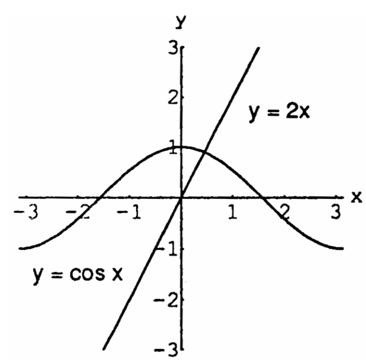
(b) $f(x) = \cos 3x - x \Rightarrow f'(x) = -3 \sin 3x - 1 \Rightarrow x_{n+1} = x_n - \frac{\cos(3x_n) - x_n}{-3\sin(3x_n) - 1}$; at approximately -0.979367, -0.887726, and 0.39004 we have $\cos 3x = x$



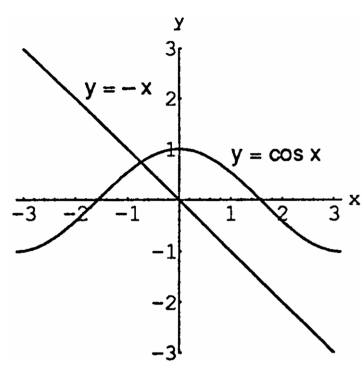
17. $f(x) = 2x^4 - 4x^2 + 1 \Rightarrow f'(x) = 8x^3 - 8x \Rightarrow x_{n+1} = x_n - \frac{2x_n^4 - 4x_n^2 + 1}{8x_n^3 - 8x_n}$; if $x_0 = -2$, then $x_6 = -1.30656296$; if $x_0 = -0.5$, then $x_3 = -0.5411961$; the roots are approximately ± 0.5411961 and ± 1.30656296 because f(x) is an even function.

18. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow x_{n+1} = x_n - \frac{\tan(x_n)}{\sec^2(x_n)}$; $x_0 = 3 \Rightarrow x_1 = 3.13971 \Rightarrow x_2 = 3.14159$ and we approximate π to be 3.14159.

19. From the graph we let $x_0 = 0.5$ and $f(x) = \cos x - 2x$
$\Rightarrow x_{n+1} = x_n - \frac{\cos(x_n) - 2x_n}{-\sin(x_n) - 2} \Rightarrow x_1 = .45063$
$\Rightarrow x_2 = .45018 \Rightarrow$ at $x \approx 0.45$ we have $\cos x = 2x$.



20. From the graph we let $x_0 = -0.7$ and $f(x) = \cos x + x$
$\Rightarrow x_{n+1} = x_n - \frac{x_n + \cos(x_n)}{1 - \sin(x_n)} \Rightarrow x_1 = -.73944$
$\Rightarrow x_2 = -.73908 \Rightarrow$ at $x \approx -0.74$ we have $\cos x = -x$.



21. The x-coordinate of the point of intersection of $y = x^2(x+1)$ and $y = \frac{1}{x}$ is the solution of $x^2(x+1) = \frac{1}{x}$
$\Rightarrow x^3 + x^2 - \frac{1}{x} = 0 \Rightarrow$ The x-coordinate is the root of $f(x) = x^3 + x^2 - \frac{1}{x} \Rightarrow f'(x) = 3x^2 + 2x + \frac{1}{x^2}$. Let $x_0 = 1$
$\Rightarrow x_{n+1} = x_n - \frac{x_n^3 + x_n^2 - \frac{1}{x_n}}{3x_n^2 + 2x_n + \frac{1}{x_n^2}} \Rightarrow x_1 = 0.83333 \Rightarrow x_2 = 0.81924 \Rightarrow x_3 = 0.81917 \Rightarrow x_7 = 0.81917 \Rightarrow r \approx 0.8192$

22. The x-coordinate of the point of intersection of $y = \sqrt{x}$ and $y = 3 - x^2$ is the solution of $\sqrt{x} = 3 - x^2$
$\Rightarrow \sqrt{x} - 3 + x^2 = 0 \Rightarrow$ The x-coordinate is the root of $f(x) = \sqrt{x} - 3 + x^2 \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} + 2x$. Let $x_0 = 1$
$\Rightarrow x_{n+1} = x_n - \frac{\sqrt{x_n} - 3 + x_n^2}{\frac{1}{2\sqrt{x_n}} + 2x_n} \Rightarrow x_1 = 1.4 \Rightarrow x_2 = 1.35556 \Rightarrow x_3 = 1.35498 \Rightarrow x_7 = 1.35498 \Rightarrow r \approx 1.3550$

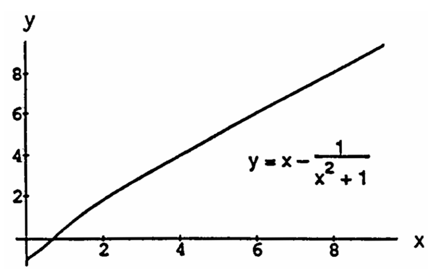
23. If $f(x) = x^3 + 2x - 4$, then $f(1) = -1 < 0$ and $f(2) = 8 > 0 \Rightarrow$ by the Intermediate Value Theorem the equation $x^3 + 2x - 4 = 0$ has a solution between 1 and 2. Consequently, $f'(x) = 3x^2 + 2$ and $x_{n+1} = x_n - \frac{x_n^3 + 2x_n - 4}{3x_n^2 + 2}$. Then $x_0 = 1 \Rightarrow x_1 = 1.2 \Rightarrow x_2 = 1.17975 \Rightarrow x_3 = 1.179509 \Rightarrow x_4 = 1.1795090 \Rightarrow$ the root is approximately 1.17951.

24. We wish to solve $8x^4 - 14x^3 - 9x^2 + 11x - 1 = 0$. Let $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$, then $f'(x) = 32x^3 - 42x^2 - 18x + 11 \Rightarrow x_{n+1} = x_n - \frac{8x_n^4 - 14x_n^3 - 9x_n^2 + 11x_n - 1}{32x_n^3 - 42x_n^2 - 18x_n + 11}$.

x_0	approximation of corresponding root
-1.0	-0.976823589
0.1	0.100363332
0.6	0.642746671
2.0	1.983713587

25. $f(x) = 4x^4 - 4x^2 \Rightarrow f'(x) = 16x^3 - 8x \Rightarrow x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 - x_i}{4x_i^2 - 2}$. Iterations are performed using the procedure in problem 13 in this section.
   (a) For $x_0 = -2$ or $x_0 = -0.8$, $x_i \to -1$ as i gets large.
   (b) For $x_0 = -0.5$ or $x_0 = 0.25$, $x_i \to 0$ as i gets large.
   (c) For $x_0 = 0.8$ or $x_0 = 2$, $x_i \to 1$ as i gets large.
   (d) (If your calculator has a CAS, put it in exact mode, otherwise approximate the radicals with a decimal value.) For $x_0 = -\frac{\sqrt{21}}{7}$ or $x_0 = -\frac{\sqrt{21}}{7}$, Newton's method does not converge. The values of x_i alternate between $x_0 = -\frac{\sqrt{21}}{7}$ or $x_0 = -\frac{\sqrt{21}}{7}$ as i increases.

26. (a) The distance can be represented by $D(x) = \sqrt{(x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2}$, where $x \geq 0$. The distance D(x) is minimized when $f(x) = (x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2$ is minimized. If $f(x) = (x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2$, then $f'(x) = 4\left(x^3 + x - 1\right)$ and $f''(x) = 4\left(3x^2 + 1\right) > 0$. Now $f'(x) = 0 \Rightarrow x^3 + x - 1 = 0 \Rightarrow x\left(x^2 + 1\right) = 1$ $\Rightarrow x = \frac{1}{x^2+1}$.



   (b) Let $g(x) = \frac{1}{x^2+1} - x = \left(x^2+1\right)^{-1} - x \Rightarrow g'(x) = -\left(x^2+1\right)^{-2}(2x) - 1 = \frac{-2x}{\left(x^2+1\right)^2} - 1$

$$\Rightarrow x_{n+1} = x_n - \frac{\left(\frac{1}{x_n^2+1} - x_n\right)}{\left(\frac{-2x_n}{\left(x_n^2+1\right)^2} - 1\right)}; \; x_0 = 1 \Rightarrow x_4 = 0.68233 \text{ to five decimal places.}$$

27. $f(x) = (x-1)^{40} \Rightarrow f'(x) = 40(x-1)^{39} \Rightarrow x_{n+1} = x_n - \frac{(x_n - 1)^{40}}{40\,(x_n - 1)^{39}} = \frac{39x_n + 1}{40}$. With $x_0 = 2$, our computer gave $x_{87} = x_{88} = x_{89} = \cdots = x_{200} = 1.11051$, coming within 0.11051 of the root $x = 1$.

28. Since $s = r\theta \Rightarrow 3 = r\theta \Rightarrow \theta = \frac{3}{r}$. Bisect the angle θ to obtain a right tringle with hypotenuse r and opposite side of length 1. Then $\sin\frac{\theta}{2} = \frac{1}{r} \Rightarrow \sin\frac{\left(\frac{3}{r}\right)}{2} = \frac{1}{r} \Rightarrow \sin\left(\frac{3}{2r}\right) = \frac{1}{r} \Rightarrow \sin\frac{3}{2r} - \frac{1}{r} = 0$. Thus the solution r is a root of $f(r) = \sin\left(\frac{3}{2r}\right) - \frac{1}{r} \Rightarrow f'(r) = -\frac{3}{2r^2}\cos\left(\frac{3}{2r}\right) + \frac{1}{r^2}$; $r_0 = 1 \Rightarrow r_{n+1} = r_n - \frac{\sin\left(\frac{3}{2r_n}\right) - \frac{1}{r_n}}{-\frac{3}{2r_n^2}\cos\left(\frac{3}{2r_n}\right) + \frac{1}{r_n^2}} \Rightarrow r_1 = 1.00280$ $\Rightarrow r_2 = 1.00282 \Rightarrow r_3 = 1.00282 \Rightarrow r \approx 1.0028 \Rightarrow \theta = \frac{3}{1.00282} \approx 2.9916$

4.7 ANTIDERIVATIVES

1. (a) x^2 (b) $\frac{x^3}{3}$ (c) $\frac{x^3}{3} - x^2 + x$

2. (a) $3x^2$ (b) $\frac{x^8}{8}$ (c) $\frac{x^8}{8} - 3x^2 + 8x$

3. (a) x^{-3} (b) $-\frac{x^{-3}}{3}$ (c) $-\frac{x^{-3}}{3} + x^2 + 3x$

4. (a) $-x^{-2}$ (b) $-\frac{x^{-2}}{4} + \frac{x^3}{3}$ (c) $\frac{x^{-2}}{2} + \frac{x^2}{2} - x$

5. (a) $\frac{-1}{x}$ (b) $\frac{-5}{x}$ (c) $2x + \frac{5}{x}$

6. (a) $\frac{1}{x^2}$ (b) $\frac{-1}{4x^2}$ (c) $\frac{x^4}{4} + \frac{1}{2x^2}$

7. (a) $\sqrt{x^3}$ (b) $\sqrt{x}$ (c) $\frac{2}{3}\sqrt{x^3} + 2\sqrt{x}$

8. (a) $x^{4/3}$ (b) $\frac{1}{2}x^{2/3}$ (c) $\frac{3}{4}x^{4/3} + \frac{3}{2}x^{2/3}$

9. (a) $x^{2/3}$ (b) $x^{1/3}$ (c) $x^{-1/3}$

10. (a) $x^{1/2}$ (b) $x^{-1/2}$ (c) $x^{-3/2}$

11. (a) $\cos(\pi x)$ (b) $-3\cos x$ (c) $\frac{-\cos(\pi x)}{\pi} + \cos(3x)$

12. (a) $\sin(\pi x)$ (b) $\sin\left(\frac{\pi x}{2}\right)$ (c) $\left(\frac{2}{\pi}\right)\sin\left(\frac{\pi x}{2}\right) + \pi\sin x$

13. (a) $\tan x$ (b) $2\tan\left(\frac{x}{3}\right)$ (c) $-\frac{2}{3}\tan\left(\frac{3x}{2}\right)$

14. (a) $-\cot x$ (b) $\cot\left(\frac{3x}{2}\right)$ (c) $x + 4\cot(2x)$

15. (a) $-\csc x$ (b) $\frac{1}{5}\csc(5x)$ (c) $2\csc\left(\frac{\pi x}{2}\right)$

16. (a) $\sec x$ (b) $\frac{4}{3}\sec(3x)$ (c) $\frac{2}{\pi}\sec\left(\frac{\pi x}{2}\right)$

17. $\int (x+1)\,dx = \frac{x^2}{2} + x + C$

18. $\int (5-6x)\,dx = 5x - 3x^2 + C$

19. $\int \left(3t^2 + \frac{t}{2}\right) dt = t^3 + \frac{t^2}{4} + C$

20. $\int \left(\frac{t^2}{2} + 4t^3\right) dt = \frac{t^3}{6} + t^4 + C$

21. $\int (2x^3 - 5x + 7)\,dx = \frac{1}{2}x^4 - \frac{5}{2}x^2 + 7x + C$

22. $\int (1 - x^2 - 3x^5)\,dx = x - \frac{1}{3}x^3 - \frac{1}{2}x^6 + C$

23. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx = \int \left(x^{-2} - x^2 - \frac{1}{3}\right) dx = \frac{x^{-1}}{-1} - \frac{x^3}{3} - \frac{1}{3}x + C = -\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C$

24. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx = \int \left(\frac{1}{5} - 2x^{-3} + 2x\right) dx = \frac{1}{5}x - \left(\frac{2x^{-2}}{-2}\right) + \frac{2x^2}{2} + C = \frac{x}{5} + \frac{1}{x^2} + x^2 + C$

25. $\int x^{-1/3}\,dx = \frac{x^{2/3}}{\frac{2}{3}} + C = \frac{3}{2}\,x^{2/3} + C$

26. $\int x^{-5/4}\,dx = \frac{x^{-1/4}}{-\frac{1}{4}} + C = \frac{-4}{\sqrt[4]{x}} + C$

27. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx = \int \left(x^{1/2} + x^{1/3}\right) dx = \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{4/3}}{\frac{4}{3}} + C = \frac{2}{3}\,x^{3/2} + \frac{3}{4}\,x^{4/3} + C$

28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx = \int \left(\frac{1}{2}\,x^{1/2} + 2x^{-1/2}\right) dx = \frac{1}{2}\left(\frac{x^{3/2}}{\frac{3}{2}}\right) + 2\left(\frac{x^{1/2}}{\frac{1}{2}}\right) + C = \frac{1}{3}\,x^{3/2} + 4x^{1/2} + C$

29. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy = \int \left(8y - 2y^{-1/4}\right) dy = \frac{8y^2}{2} - 2\left(\frac{y^{3/4}}{\frac{3}{4}}\right) + C = 4y^2 - \frac{8}{3}\,y^{3/4} + C$

30. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy = \int \left(\frac{1}{7} - y^{-5/4}\right) dy = \frac{1}{7}\,y - \left(\frac{y^{-1/4}}{-\frac{1}{4}}\right) + C = \frac{y}{7} + \frac{4}{y^{1/4}} + C$

31. $\int 2x\,(1 - x^{-3})\,dx = \int (2x - 2x^{-2})\,dx = \frac{2x^2}{2} - 2\left(\frac{x^{-1}}{-1}\right) + C = x^2 + \frac{2}{x} + C$

32. $\int x^{-3}\,(x + 1)\,dx = \int (x^{-2} + x^{-3})\,dx = \frac{x^{-1}}{-1} + \left(\frac{x^{-2}}{-2}\right) + C = -\frac{1}{x} - \frac{1}{2x^2} + C$

33. $\int \frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt = \int \left(\frac{t^{3/2}}{t^2} + \frac{t^{1/2}}{t^2}\right) dt = \int \left(t^{-1/2} + t^{-3/2}\right) dt = \frac{t^{1/2}}{\frac{1}{2}} + \left(\frac{t^{-1/2}}{-\frac{1}{2}}\right) + C = 2\sqrt{t} - \frac{2}{\sqrt{t}} + C$

34. $\int \frac{4+\sqrt{t}}{t^3}\,dt = \int \left(\frac{4}{t^3} + \frac{t^{1/2}}{t^3}\right) dt = \int \left(4t^{-3} + t^{-5/2}\right) dt = 4\left(\frac{t^{-2}}{-2}\right) + \left(\frac{t^{-3/2}}{-\frac{3}{2}}\right) + C = -\frac{2}{t^2} - \frac{2}{3t^{3/2}} + C$

35. $\int -2\cos t\,dt = -2\sin t + C$

36. $\int -5\sin t\,dt = 5\cos t + C$

37. $\int 7\sin\frac{\theta}{3}\,d\theta = -21\cos\frac{\theta}{3} + C$

38. $\int 3\cos 5\theta\,d\theta = \frac{3}{5}\sin 5\theta + C$

39. $\int -3\csc^2 x\,dx = 3\cot x + C$

40. $\int -\frac{\sec^2 x}{3}\,dx = -\frac{\tan x}{3} + C$

41. $\int \frac{\csc\theta\cot\theta}{2}\,d\theta = -\frac{1}{2}\csc\theta + C$

42. $\int \frac{2}{5}\sec\theta\tan\theta\,d\theta = \frac{2}{5}\sec\theta + C$

43. $\int (4\sec x\tan x - 2\sec^2 x)\,dx = 4\sec x - 2\tan x + C$

44. $\int \frac{1}{2}(\csc^2 x - \csc x\cot x)\,dx = -\frac{1}{2}\cot x + \frac{1}{2}\csc x + C$

45. $\int (\sin 2x - \csc^2 x)\,dx = -\frac{1}{2}\cos 2x + \cot x + C$

46. $\int (2\cos 2x - 3\sin 3x)\,dx = \sin 2x + \cos 3x + C$

47. $\int \frac{1+\cos 4t}{2}\,dt = \int \left(\frac{1}{2} + \frac{1}{2}\cos 4t\right) dt = \frac{1}{2}\,t + \frac{1}{2}\left(\frac{\sin 4t}{4}\right) + C = \frac{t}{2} + \frac{\sin 4t}{8} + C$

48. $\int \frac{1-\cos 6t}{2}\,dt = \int \left(\frac{1}{2} - \frac{1}{2}\cos 6t\right) dt = \frac{1}{2}\,t - \frac{1}{2}\left(\frac{\sin 6t}{6}\right) + C = \frac{t}{2} - \frac{\sin 6t}{12} + C$

49. $\int (1 + \tan^2\theta)\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C$

50. $\int (2 + \tan^2\theta)\,d\theta = \int (1 + 1 + \tan^2\theta)\,d\theta = \int (1 + \sec^2\theta)\,d\theta = \theta + \tan\theta + C$

51. $\int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx = -\cot x - x + C$

52. $\int (1 - \cot^2 x)\, dx = \int (1 - (\csc^2 x - 1))\, dx = \int (2 - \csc^2 x)\, dx = 2x + \cot x + C$

53. $\int \cos\theta (\tan\theta + \sec\theta)\, d\theta = \int (\sin\theta + 1)\, d\theta = -\cos\theta + \theta + C$

54. $\int \frac{\csc\theta}{\csc\theta - \sin\theta}\, d\theta = \int \left(\frac{\csc\theta}{\csc\theta - \sin\theta}\right)\left(\frac{\sin\theta}{\sin\theta}\right) d\theta = \int \frac{1}{1-\sin^2\theta}\, d\theta = \int \frac{1}{\cos^2\theta}\, d\theta = \int \sec^2\theta\, d\theta = \tan\theta + C$

55. $\frac{d}{dx}\left(\frac{(7x-2)^4}{28} + C\right) = \frac{4(7x-2)^3(7)}{28} = (7x-2)^3$

56. $\frac{d}{dx}\left(-\frac{(3x+5)^{-1}}{3} + C\right) = -\left(-\frac{(3x+5)^{-2}(3)}{3}\right) = (3x+5)^{-2}$

57. $\frac{d}{dx}\left(\frac{1}{5}\tan(5x-1) + C\right) = \frac{1}{5}\left(\sec^2(5x-1)\right)(5) = \sec^2(5x-1)$

58. $\frac{d}{dx}\left(-3\cot\left(\frac{x-1}{3}\right) + C\right) = -3\left(-\csc^2\left(\frac{x-1}{3}\right)\right)\left(\frac{1}{3}\right) = \csc^2\left(\frac{x-1}{3}\right)$

59. $\frac{d}{dx}\left(\frac{-1}{x+1} + C\right) = (-1)(-1)(x+1)^{-2} = \frac{1}{(x+1)^2}$

60. $\frac{d}{dx}\left(\frac{x}{x+1} + C\right) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$

61. (a) Wrong: $\frac{d}{dx}\left(\frac{x^2}{2}\sin x + C\right) = \frac{2x}{2}\sin x + \frac{x^2}{2}\cos x = x\sin x + \frac{x^2}{2}\cos x \neq x\sin x$
(b) Wrong: $\frac{d}{dx}(-x\cos x + C) = -\cos x + x\sin x \neq x\sin x$
(c) Right: $\frac{d}{dx}(-x\cos x + \sin x + C) = -\cos x + x\sin x + \cos x = x\sin x$

62. (a) Wrong: $\frac{d}{d\theta}\left(\frac{\sec^3\theta}{3} + C\right) = \frac{3\sec^2\theta}{3}(\sec\theta\tan\theta) = \sec^3\theta\tan\theta \neq \tan\theta\sec^2\theta$
(b) Right: $\frac{d}{d\theta}\left(\frac{1}{2}\tan^2\theta + C\right) = \frac{1}{2}(2\tan\theta)\sec^2\theta = \tan\theta\sec^2\theta$
(c) Right: $\frac{d}{d\theta}\left(\frac{1}{2}\sec^2\theta + C\right) = \frac{1}{2}(2\sec\theta)\sec\theta\tan\theta = \tan\theta\sec^2\theta$

63. (a) Wrong: $\frac{d}{dx}\left(\frac{(2x+1)^3}{3} + C\right) = \frac{3(2x+1)^2(2)}{3} = 2(2x+1)^2 \neq (2x+1)^2$
(b) Wrong: $\frac{d}{dx}\left((2x+1)^3 + C\right) = 3(2x+1)^2(2) = 6(2x+1)^2 \neq 3(2x+1)^2$
(c) Right: $\frac{d}{dx}\left((2x+1)^3 + C\right) = 6(2x+1)^2$

64. (a) Wrong: $\frac{d}{dx}(x^2+x+C)^{1/2} = \frac{1}{2}(x^2+x+C)^{-1/2}(2x+1) = \frac{2x+1}{2\sqrt{x^2+x+C}} \neq \sqrt{2x+1}$
(b) Wrong: $\frac{d}{dx}\left((x^2+x)^{1/2} + C\right) = \frac{1}{2}(x^2+x)^{-1/2}(2x+1) = \frac{2x+1}{2\sqrt{x^2+x}} \neq \sqrt{2x+1}$
(c) Right: $\frac{d}{dx}\left(\frac{1}{3}\left(\sqrt{2x+1}\right)^3 + C\right) = \frac{d}{dx}\left(\frac{1}{3}(2x+1)^{3/2} + C\right) = \frac{3}{6}(2x+1)^{1/2}(2) = \sqrt{2x+1}$

65. Right: $\frac{d}{dx}\left(\left(\frac{x+3}{x-2}\right)^3 + C\right) = 3\left(\frac{x+3}{x-2}\right)^2 \frac{(x-2)\cdot 1 - (x+3)\cdot 1}{(x-2)^2} = 3\frac{(x+3)^2}{(x-2)^2}\frac{-5}{(x-2)^2} = \frac{-15(x+3)^2}{(x-2)^4}$

66. Wrong: $\frac{d}{dx}\left(\frac{\sin(x^2)}{x} + C\right) = \frac{x\cdot\cos(x^2)(2x) - \sin(x^2)\cdot 1}{x^2} = \frac{2x^2\cos(x^2) - \sin(x^2)}{x^2} \neq \frac{x\cos(x^2) - \sin(x^2)}{x^2}$

67. Graph (b), because $\frac{dy}{dx} = 2x \Rightarrow y = x^2 + C$. Then $y(1) = 4 \Rightarrow C = 3$.

68. Graph (b), because $\frac{dy}{dx} = -x \Rightarrow y = -\frac{1}{2}x^2 + C$. Then $y(-1) = 1 \Rightarrow C = \frac{3}{2}$.

69. $\frac{dy}{dx} = 2x - 7 \Rightarrow y = x^2 - 7x + C$; at $x = 2$ and $y = 0$ we have $0 = 2^2 - 7(2) + C \Rightarrow C = 10 \Rightarrow y = x^2 - 7x + 10$

70. $\frac{dy}{dx} = 10 - x \Rightarrow y = 10x - \frac{x^2}{2} + C$; at $x = 0$ and $y = -1$ we have $-1 = 10(0) - \frac{0^2}{2} + C \Rightarrow C = -1 \Rightarrow y = 10x - \frac{x^2}{2} - 1$

71. $\frac{dy}{dx} = \frac{1}{x^2} + x = x^{-2} + x \Rightarrow y = -x^{-1} + \frac{x^2}{2} + C$; at $x = 2$ and $y = 1$ we have $1 = -2^{-1} + \frac{2^2}{2} + C \Rightarrow C = -\frac{1}{2}$
$\Rightarrow y = -x^{-1} + \frac{x^2}{2} - \frac{1}{2}$ or $y = -\frac{1}{x} + \frac{x^2}{2} - \frac{1}{2}$

72. $\frac{dy}{dx} = 9x^2 - 4x + 5 \Rightarrow y = 3x^3 - 2x^2 + 5x + C$; at $x = -1$ and $y = 0$ we have $0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + C$
$\Rightarrow C = 10 \Rightarrow y = 3x^3 - 2x^2 + 5x + 10$

73. $\frac{dy}{dx} = 3x^{-2/3} \Rightarrow y = \frac{3x^{1/3}}{\frac{1}{3}} + C = 9$; at $x = 9x^{1/3} + C$; at $x = -1$ and $y = -5$ we have $-5 = 9(-1)^{1/3} + C \Rightarrow C = 4$
$\Rightarrow y = 9x^{1/3} + 4$

74. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2} \Rightarrow y = x^{1/2} + C$; at $x = 4$ and $y = 0$ we have $0 = 4^{1/2} + C \Rightarrow C = -2 \Rightarrow y = x^{1/2} - 2$

75. $\frac{ds}{dt} = 1 + \cos t \Rightarrow s = t + \sin t + C$; at $t = 0$ and $s = 4$ we have $4 = 0 + \sin 0 + C \Rightarrow C = 4 \Rightarrow s = t + \sin t + 4$

76. $\frac{ds}{dt} = \cos t + \sin t \Rightarrow s = \sin t - \cos t + C$; at $t = \pi$ and $s = 1$ we have $1 = \sin \pi - \cos \pi + C \Rightarrow C = 0$
$\Rightarrow s = \sin t - \cos t$

77. $\frac{dr}{d\theta} = -\pi \sin \pi\theta \Rightarrow r = \cos(\pi\theta) + C$; at $r = 0$ and $\theta = 0$ we have $0 = \cos(\pi 0) + C \Rightarrow C = -1 \Rightarrow r = \cos(\pi\theta) - 1$

78. $\frac{dr}{d\theta} = \cos \pi\theta \Rightarrow r = \frac{1}{\pi}\sin(\pi\theta) + C$; at $r = 1$ and $\theta = 0$ we have $1 = \frac{1}{\pi}\sin(\pi 0) + C \Rightarrow C = 1 \Rightarrow r = \frac{1}{\pi}\sin(\pi\theta) + 1$

79. $\frac{dv}{dt} = \frac{1}{2}\sec t \tan t \Rightarrow v = \frac{1}{2}\sec t + C$; at $v = 1$ and $t = 0$ we have $1 = \frac{1}{2}\sec(0) + C \Rightarrow C = \frac{1}{2} \Rightarrow v = \frac{1}{2}\sec t + \frac{1}{2}$

80. $\frac{dv}{dt} = 8t + \csc^2 t \Rightarrow v = 4t^2 - \cot t + C$; at $v = -7$ and $t = \frac{\pi}{2}$ we have $-7 = 4\left(\frac{\pi}{2}\right)^2 - \cot\left(\frac{\pi}{2}\right) + C \Rightarrow C = -7 - \pi^2$
$\Rightarrow v = 4t^2 - \cot t - 7 - \pi^2$

81. $\frac{d^2y}{dx^2} = 2 - 6x \Rightarrow \frac{dy}{dx} = 2x - 3x^2 + C_1$; at $\frac{dy}{dx} = 4$ and $x = 0$ we have $4 = 2(0) - 3(0)^2 + C_1 \Rightarrow C_1 = 4$
$\Rightarrow \frac{dy}{dx} = 2x - 3x^2 + 4 \Rightarrow y = x^2 - x^3 + 4x + C_2$; at $y = 1$ and $x = 0$ we have $1 = 0^2 - 0^3 + 4(0) + C_2 \Rightarrow C_2 = 1$
$\Rightarrow y = x^2 - x^3 + 4x + 1$

82. $\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{dy}{dx} = C_1$; at $\frac{dy}{dx} = 2$ and $x = 0$ we have $C_1 = 2 \Rightarrow \frac{dy}{dx} = 2 \Rightarrow y = 2x + C_2$; at $y = 0$ and $x = 0$ we have $0 = 2(0) + C_2 \Rightarrow C_2 = 0 \Rightarrow y = 2x$

83. $\frac{d^2r}{dt^2} = \frac{2}{t^3} = 2t^{-3} \Rightarrow \frac{dr}{dt} = -t^{-2} + C_1$; at $\frac{dr}{dt} = 1$ and $t = 1$ we have $1 = -(1)^{-2} + C_1 \Rightarrow C_1 = 2 \Rightarrow \frac{dr}{dt} = -t^{-2} + 2$
$\Rightarrow r = t^{-1} + 2t + C_2$; at $r = 1$ and $t = 1$ we have $1 = 1^{-1} + 2(1) + C_2 \Rightarrow C_2 = -2 \Rightarrow r = t^{-1} + 2t - 2$ or
$r = \frac{1}{t} + 2t - 2$

84. $\frac{d^2s}{dt^2} = \frac{3t}{8} \Rightarrow \frac{ds}{dt} = \frac{3t^2}{16} + C_1$; at $\frac{ds}{dt} = 3$ and $t = 4$ we have $3 = \frac{3(4)^2}{16} + C_1 \Rightarrow C_1 = 0 \Rightarrow \frac{ds}{dt} = \frac{3t^2}{16} \Rightarrow s = \frac{t^3}{16} + C_2$; at $s = 4$ and $t = 4$ we have $4 = \frac{4^3}{16} + C_2 \Rightarrow C_2 = 0 \Rightarrow s = \frac{t^3}{16}$

85. $\frac{d^3y}{dx^3} = 6 \Rightarrow \frac{d^2y}{dx^2} = 6x + C_1$; at $\frac{d^2y}{dx^2} = -8$ and $x = 0$ we have $-8 = 6(0) + C_1 \Rightarrow C_1 = -8 \Rightarrow \frac{d^2y}{dx^2} = 6x - 8$ $\Rightarrow \frac{dy}{dx} = 3x^2 - 8x + C_2$; at $\frac{dy}{dx} = 0$ and $x = 0$ we have $0 = 3(0)^2 - 8(0) + C_2 \Rightarrow C_2 = 0 \Rightarrow \frac{dy}{dx} = 3x^2 - 8x$ $\Rightarrow y = x^3 - 4x^2 + C_3$; at $y = 5$ and $x = 0$ we have $5 = 0^3 - 4(0)^2 + C_3 \Rightarrow C_3 = 5 \Rightarrow y = x^3 - 4x^2 + 5$

86. $\frac{d^3\theta}{dt^3} = 0 \Rightarrow \frac{d^2\theta}{dt^2} = C_1$; at $\frac{d^2\theta}{dt^2} = -2$ and $t = 0$ we have $\frac{d^2\theta}{dt^2} = -2 \Rightarrow \frac{d\theta}{dt} = -2t + C_2$; at $\frac{d\theta}{dt} = -\frac{1}{2}$ and $t = 0$ we have $-\frac{1}{2} = -2(0) + C_2 \Rightarrow C_2 = -\frac{1}{2} \Rightarrow \frac{d\theta}{dt} = -2t - \frac{1}{2} \Rightarrow \theta = -t^2 - \frac{1}{2}t + C_3$; at $\theta = \sqrt{2}$ and $t = 0$ we have $\sqrt{2} = -0^2 - \frac{1}{2}(0) + C_3 \Rightarrow C_3 = \sqrt{2} \Rightarrow \theta = -t^2 - \frac{1}{2}t + \sqrt{2}$

87. $y^{(4)} = -\sin t + \cos t \Rightarrow y''' = \cos t + \sin t + C_1$; at $y''' = 7$ and $t = 0$ we have $7 = \cos(0) + \sin(0) + C_1 \Rightarrow C_1 = 6$ $\Rightarrow y''' = \cos t + \sin t + 6 \Rightarrow y'' = \sin t - \cos t + 6t + C_2$; at $y'' = -1$ and $t = 0$ we have $-1 = \sin(0) - \cos(0) + 6(0) + C_2 \Rightarrow C_2 = 0 \Rightarrow y'' = \sin t - \cos t + 6t \Rightarrow y' = -\cos t - \sin t + 3t^2 + C_3$; at $y' = -1$ and $t = 0$ we have $-1 = -\cos(0) - \sin(0) + 3(0)^2 + C_3 \Rightarrow C_3 = 0 \Rightarrow y' = -\cos t - \sin t + 3t^2$ $\Rightarrow y = -\sin t + \cos t + t^3 + C_4$; at $y = 0$ and $t = 0$ we have $0 = -\sin(0) + \cos(0) + 0^3 + C_4 \Rightarrow C_4 = -1$ $\Rightarrow y = -\sin t + \cos t + t^3 - 1$

88. $y^{(4)} = -\cos x + 8\sin(2x) \Rightarrow y''' = -\sin x - 4\cos(2x) + C_1$; at $y''' = 0$ and $x = 0$ we have $0 = -\sin(0) - 4\cos(2(0)) + C_1 \Rightarrow C_1 = 4 \Rightarrow y''' = -\sin x - 4\cos(2x) + 4 \Rightarrow y'' = \cos x - 2\sin(2x) + 4x + C_2$; at $y'' = 1$ and $x = 0$ we have $1 = \cos(0) - 2\sin(2(0)) + 4(0) + C_2 \Rightarrow C_2 = 0 \Rightarrow y'' = \cos x - 2\sin(2x) + 4x$ $\Rightarrow y' = \sin x + \cos(2x) + 2x^2 + C_3$; at $y' = 1$ and $x = 0$ we have $1 = \sin(0) + \cos(2(0)) + 2(0)^2 + C_3 \Rightarrow C_3 = 0$ $\Rightarrow y' = \sin x + \cos(2x) + 2x^2 \Rightarrow y = -\cos x + \frac{1}{2}\sin(2x) + \frac{2}{3}x^3 + C_4$; at $y = 3$ and $x = 0$ we have $3 = -\cos(0) + \frac{1}{2}\sin(2(0)) + \frac{2}{3}(0)^3 + C_4 \Rightarrow C_4 = 4 \Rightarrow y = -\cos x + \frac{1}{2}\sin(2x) + \frac{2}{3}x^3 + 4$

89. $m = y' = 3\sqrt{x} = 3x^{1/2} \Rightarrow y = 2x^{3/2} + C$; at $(9, 4)$ we have $4 = 2(9)^{3/2} + C \Rightarrow C = -50 \Rightarrow y = 2x^{3/2} - 50$

90. Yes. If $F(x)$ and $G(x)$ both solve the initial value problem on an interval I then they both have the same first derivative. Therefore, by Corollary 2 of the Mean Value Theorem there is a constant C such that $F(x) = G(x) + C$ for all x. In particular, $F(x_0) = G(x_0) + C$, so $C = F(x_0) - G(x_0) = 0$. Hence $F(x) = G(x)$ for all x.

91. $\frac{dy}{dx} = 1 - \frac{4}{3}x^{1/3} \Rightarrow y = \int \left(1 - \frac{4}{3}x^{1/3}\right) dx = x - x^{4/3} + C$; at $(1, 0.5)$ on the curve we have $0.5 = 1 - 1^{4/3} + C$ $\Rightarrow C = 0.5 \Rightarrow y = x - x^{4/3} + \frac{1}{2}$

92. $\frac{dy}{dx} = x - 1 \Rightarrow y = \int (x - 1)\, dx = \frac{x^2}{2} - x + C$; at $(-1, 1)$ on the curve we have $1 = \frac{(-1)^2}{2} - (-1) + C \Rightarrow C = -\frac{1}{2}$ $\Rightarrow y = \frac{x^2}{2} - x - \frac{1}{2}$

93. $\frac{dy}{dx} = \sin x - \cos x \Rightarrow y = \int (\sin x - \cos x)\, dx = -\cos x - \sin x + C$; at $(-\pi, -1)$ on the curve we have $-1 = -\cos(-\pi) - \sin(-\pi) + C \Rightarrow C = -2 \Rightarrow y = -\cos x - \sin x - 2$

94. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \pi \sin \pi x = \frac{1}{2}x^{-1/2} + \pi \sin \pi x \Rightarrow y = \int \left(\frac{1}{2}x^{-1/2} + \sin \pi x\right) dx = x^{1/2} - \cos \pi x + C$; at $(1, 2)$ on the curve we have $2 = 1^{1/2} - \cos \pi(1) + C \Rightarrow C = 0 \Rightarrow y = \sqrt{x} - \cos \pi x$

95. (a) $\frac{ds}{dt} = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C$; (i) at $s = 5$ and $t = 0$ we have $C = 5 \Rightarrow s = 4.9t^2 - 3t + 5$; displacement $= s(3) - s(1) = ((4.9)(9) - 9 + 5) - (4.9 - 3 + 5) = 33.2$ units; (ii) at $s = -2$ and $t = 0$ we have $C = -2 \Rightarrow s = 4.9t^2 - 3t - 2$; displacement $= s(3) - s(1) = ((4.9)(9) - 9 - 2) - (4.9 - 3 - 2) = 33.2$ units; (iii) at $s = s_0$ and $t = 0$ we have $C = s_0 \Rightarrow s = 4.9t^2 - 3t + s_0$; displacement $= s(3) - s(1)$ $= ((4.9)(9) - 9 + s_0) - (4.9 - 3 + s_0) = 33.2$ units

(b) True. Given an antiderivative f(t) of the velocity function, we know that the body's position function is $s = f(t) + C$ for some constant C. Therefore, the displacement from $t = a$ to $t = b$ is $(f(b) + C) - (f(a) + C)$ $= f(b) - f(a)$. Thus we can find the displacement from any antiderivative f as the numerical difference $f(b) - f(a)$ without knowing the exact values of C and s.

96. $a(t) = v'(t) = 20 \Rightarrow v(t) = 20t + C$; at $(0, 0)$ we have $C = 0 \Rightarrow v(t) = 20t$. When $t = 60$, then $v(60) = 20(60) = 1200\ \frac{m}{sec}$.

97. Step 1: $\frac{d^2s}{dt^2} = -k \Rightarrow \frac{ds}{dt} = -kt + C_1$; at $\frac{ds}{dt} = 88$ and $t = 0$ we have $C_1 = 88 \Rightarrow \frac{ds}{dt} = -kt + 88 \Rightarrow$
$s = -k\left(\frac{t^2}{2}\right) + 88t + C_2$; at $s = 0$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = -\frac{kt^2}{2} + 88t$

Step 2: $\frac{ds}{dt} = 0 \Rightarrow 0 = -kt + 88 \Rightarrow t = \frac{88}{k}$

Step 3: $242 = \frac{-k\left(\frac{88}{k}\right)^2}{2} + 88\left(\frac{88}{k}\right) \Rightarrow 242 = -\frac{(88)^2}{2k} + \frac{(88)^2}{k} \Rightarrow 242 = \frac{(88)^2}{2k} \Rightarrow k = 16$

98. $\frac{d^2s}{dt^2} = -k \Rightarrow \frac{ds}{dt} = \int -k\,dt = -kt + C$; at $\frac{ds}{dt} = 44$ when $t = 0$ we have $44 = -k(0) + C \Rightarrow C = 44$
$\Rightarrow \frac{ds}{dt} = -kt + 44 \Rightarrow s = -\frac{kt^2}{2} + 44t + C_1$; at $s = 0$ when $t = 0$ we have $0 = -\frac{k(0)^2}{2} + 44(0) + C_1 \Rightarrow C_1 = 0$
$\Rightarrow s = -\frac{kt^2}{2} + 44t$. Then $\frac{ds}{dt} = 0 \Rightarrow -kt + 44 = 0 \Rightarrow t = \frac{44}{k}$ and $s\left(\frac{44}{k}\right) = -\frac{k\left(\frac{44}{k}\right)^2}{2} + 44\left(\frac{44}{k}\right) = 45$
$\Rightarrow -\frac{968}{k} + \frac{1936}{k} = 45 \Rightarrow \frac{968}{k} = 45 \Rightarrow k = \frac{968}{45} \approx 21.5\ \frac{ft}{sec^2}$.

99. (a) $v = \int a\,dt = \int \left(15t^{1/2} - 3t^{-1/2}\right) dt = 10t^{3/2} - 6t^{1/2} + C$; $\frac{ds}{dt}(1) = 4 \Rightarrow 4 = 10(1)^{3/2} - 6(1)^{1/2} + C \Rightarrow C = 0$
$\Rightarrow v = 10t^{3/2} - 6t^{1/2}$

(b) $s = \int v\,dt = \int \left(10t^{3/2} - 6t^{1/2}\right) dt = 4t^{5/2} - 4t^{3/2} + C$; $s(1) = 0 \Rightarrow 0 = 4(1)^{5/2} - 4(1)^{3/2} + C \Rightarrow C = 0$
$\Rightarrow s = 4t^{5/2} - 4t^{3/2}$

100. $\frac{d^2s}{dt^2} = -5.2 \Rightarrow \frac{ds}{dt} = -5.2t + C_1$; at $\frac{ds}{dt} = 0$ and $t = 0$ we have $C_1 = 0 \Rightarrow \frac{ds}{dt} = -5.2t \Rightarrow s = -2.6t^2 + C_2$; at $s = 4$ and $t = 0$ we have $C_2 = 4 \Rightarrow s = -2.6t^2 + 4$. Then $s = 0 \Rightarrow 0 = -2.6t^2 + 4 \Rightarrow t = \sqrt{\frac{4}{2.6}} \approx 1.24$ sec, since $t > 0$

101. $\frac{d^2s}{dt^2} = a \Rightarrow \frac{ds}{dt} = \int a\,dt = at + C$; $\frac{ds}{dt} = v_0$ when $t = 0 \Rightarrow C = v_0 \Rightarrow \frac{ds}{dt} = at + v_0 \Rightarrow s = \frac{at^2}{2} + v_0t + C_1$; $s = s_0$ when $t = 0 \Rightarrow s_0 = \frac{a(0)^2}{2} + v_0(0) + C_1 \Rightarrow C_1 = s_0 \Rightarrow s = \frac{at^2}{2} + v_0t + s_0$

102. The appropriate initial value problem is: Differential Equation: $\frac{d^2s}{dt^2} = -g$ with Initial Conditions: $\frac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$. Thus, $\frac{ds}{dt} = \int -g\,dt = -gt + C_1$; $\frac{ds}{dt}(0) = v_0 \Rightarrow v_0 = (-g)(0) + C_1 \Rightarrow C_1 = v_0$
$\Rightarrow \frac{ds}{dt} = -gt + v_0$. Thus $s = \int (-gt + v_0)\,dt = -\frac{1}{2}gt^2 + v_0t + C_2$; $s(0) = s_0 = -\frac{1}{2}(g)(0)^2 + v_0(0) + C_2 \Rightarrow C_2 = s_0$
Thus $s = -\frac{1}{2}gt^2 + v_0t + s_0$.

103 – 106 Example CAS commands:

Maple:

```
with(student):
f := x -> cos(x)^2 + sin(x);
ic := [x=Pi,y=1];
F := unapply( int( f(x), x ) + C, x );
eq := eval( y=F(x), ic );
solnC := solve( eq, {C} );
Y := unapply( eval( F(x), solnC ), x );
DEplot( diff(y(x),x) = f(x), y(x), x=0..2*Pi, [[y(Pi)=1]],
    color=black, linecolor=black, stepsize=0.05, title="Section 4.7 #103" );
```

Mathematica: (functions and values may vary)
The following commands use the definite integral and the Fundamental Theorem of calculus to construct the solution of the initial value problems for exercises 103 - 105.

Clear[x, y, yprime]
yprime[x_] = Cos[x]^2 + Sin[x];
initxvalue = π; inityvalue = 1;
y[x_] = Integrate[yprime[t], {t, initxvalue, x}] + inityvalue

If the solution satisfies the differential equation and initial condition, the following yield True

yprime[x]==D[y[x], x] //Simplify
y[initxvalue]==inityvalue

Since exercise 106 is a second order differential equation, two integrations will be required.

Clear[x, y, yprime]
y2prime[x_] = 3 Exp[x/2] + 1;
initxval = 0; inityval = 4; inityprimeval = -1;
yprime[x_] = Integrate[y2prime[t],{t, initxval, x}] + inityprimeval
y[x_] = Integrate[yprime[t], {t, initxval, x}] + inityval

Verify that y[x] solves the differential equation and initial condition and plot the solution (red) and its derivative (blue).

y2prime[x]==D[y[x], {x, 2}]//Simplify
y[initxval]==inityval
yprime[initxval]==inityprimeval
Plot[{y[x], yprime[x]}, {x, initxval $-$ 3, initxval + 3}, PlotStyle $\rightarrow$ {RGBColor[1,0,0], RGBColor[0,0,1]}]

CHAPTER 4 PRACTICE EXERCISES

1. No, since $f(x) = x^3 + 2x + \tan x \Rightarrow f'(x) = 3x^2 + 2 + \sec^2 x > 0 \Rightarrow$ f(x) is always increasing on its domain

2. No, since $g(x) = \csc x + 2 \cot x \Rightarrow g'(x) = -\csc x \cot x - 2\csc^2 x = -\frac{\cos x}{\sin^2 x} - \frac{2}{\sin^2 x} = -\frac{1}{\sin^2 x}(\cos x + 2) < 0$
$\Rightarrow$ g(x) is always decreasing on its domain

3. No absolute minimum because $\lim\limits_{x \to \infty} (7 + x)(11 - 3x)^{1/3} = -\infty$. Next $f'(x) =$
$(11 - 3x)^{1/3} - (7 + x)(11 - 3x)^{-2/3} = \frac{(11-3x)-(7+x)}{(11-3x)^{2/3}} = \frac{4(1-x)}{(11-3x)^{2/3}} \Rightarrow x = 1$ and $x = \frac{11}{3}$ are critical points.
Since $f' > 0$ if $x < 1$ and $f' < 0$ if $x > 1$, $f(1) = 16$ is the absolute maximum.

4. $f(x) = \frac{ax+b}{x^2-1} \Rightarrow f'(x) = \frac{a(x^2-1)-2x(ax+b)}{(x^2-1)^2} = \frac{-(ax^2+2bx+a)}{(x^2-1)^2}$; $f'(3) = 0 \Rightarrow -\frac{1}{64}(9a + 6b + a) = 0 \Rightarrow 5a + 3b = 0$.
We require also that $f(3) = 1$. Thus $1 = \frac{3a+b}{8} \Rightarrow 3a + b = 8$. Solving both equations yields $a = 6$ and $b = -10$. Now,
$f'(x) = \frac{-2(3x-1)(x-3)}{(x^2-1)^2}$ so that $f' = \underset{-1}{---|}\underset{1/3}{---|}\underset{1}{+++|}\underset{3}{+++|}---$. Thus f' changes sign at $x = 3$ from positive to negative so there is a local maximum at $x = 3$ which has a value $f(3) = 1$.

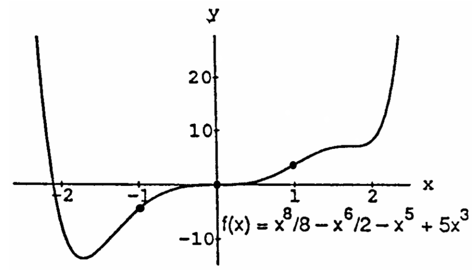
5. Yes, because at each point of $[0, 1)$ except $x = 0$, the function's value is a local minimum value as well as a local maximum value. At $x = 0$ the function's value, 0, is not a local minimum value because each open interval around $x = 0$ on the x-axis contains points to the left of 0 where f equals -1.

6. (a) The first derivative of the function $f(x) = x^3$ is zero at $x = 0$ even though f has no local extreme value at $x = 0$.
(b) Theorem 2 says only that if f is differentiable and f has a local extreme at $x = c$ then $f'(c) = 0$. It does not assert the (false) reverse implication $f'(c) = 0 \Rightarrow$ f has a local extreme at $x = c$.

7. No, because the interval $0 < x < 1$ fails to be closed. The Extreme Value Theorem says that if the function is continuous throughout a finite closed interval $a \le x \le b$ then the existence of absolute extrema is guaranteed on that interval.

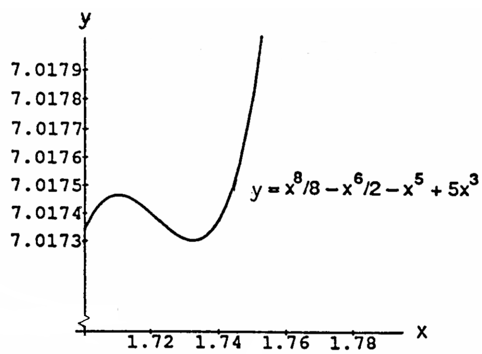
8. The absolute maximum is $|-1| = 1$ and the absolute minimum is $|0| = 0$. This is not inconsistent with the Extreme Value Theorem for continuous functions, which says a continuous function on a closed interval attains its extreme values on that interval. The theorem says nothing about the behavior of a continuous function on an interval which is half open and half closed, such as $[-1, 1)$, so there is nothing to contradict.

9. (a) There appear to be local minima at $x = -1.75$ and 1.8. Points of inflection are indicated at approximately $x = 0$ and $x = \pm 1$.

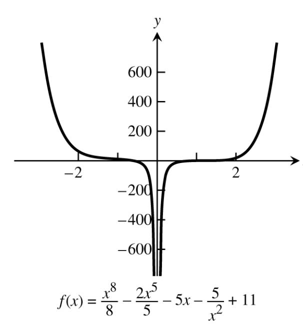


(b) $f'(x) = x^7 - 3x^5 - 5x^4 + 15x^2 = x^2(x^2 - 3)(x^3 - 5)$. The pattern $y' = \underset{-\sqrt{3}}{---\,|}\,\underset{0}{+++\,|}\,\underset{\sqrt[3]{5}}{+++\,|}\,\underset{\sqrt{3}}{---\,|}\,+++$ indicates a local maximum at $x = \sqrt[3]{5}$ and local minima at $x = \pm\sqrt{3}$.

(c)

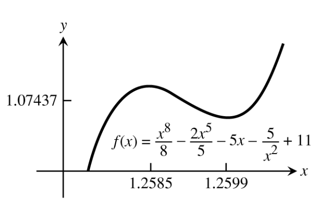


10. (a) The graph does not indicate any local extremum. Points of inflection are indicated at approximately $x = -\frac{3}{4}$ and $x = 1$.



(b) $f'(x) = x^7 - 2x^4 - 5 + \frac{10}{x^3} = x^{-3}(x^3 - 2)(x^7 - 5)$. The pattern $f' = \underset{0}{---\,)(}\,\underset{\sqrt[7]{5}}{+++\,|}\,\underset{\sqrt[3]{2}}{---\,|}\,+++$ indicates a local maximum at $x = \sqrt[7]{5}$ and a local minimum at $x = \sqrt[3]{2}$.

(c)



11. (a) $g(t) = \sin^2 t - 3t \Rightarrow g'(t) = 2 \sin t \cos t - 3 = \sin(2t) - 3 \Rightarrow g' < 0 \Rightarrow g(t)$ is always falling and hence must decrease on every interval in its domain.
(b) One, since $\sin^2 t - 3t - 5 = 0$ and $\sin^2 t - 3t = 5$ have the same solutions: $f(t) = \sin^2 t - 3t - 5$ has the same derivative as $g(t)$ in part (a) and is always decreasing with $f(-3) > 0$ and $f(0) < 0$. The Intermediate Value Theorem guarantees the continuous function f has a root in $[-3, 0]$.

12. (a) $y = \tan\theta \Rightarrow \frac{dy}{d\theta} = \sec^2\theta > 0 \Rightarrow y = \tan\theta$ is always rising on its domain $\Rightarrow y = \tan\theta$ increases on every interval in its domain
(b) The interval $\left[\frac{\pi}{4}, \pi\right]$ is not in the tangent's domain because $\tan\theta$ is undefined at $\theta = \frac{\pi}{2}$. Thus the tangent need not increase on this interval.

13. (a) $f(x) = x^4 + 2x^2 - 2 \Rightarrow f'(x) = 4x^3 + 4x$. Since $f(0) = -2 < 0$, $f(1) = 1 > 0$ and $f'(x) \ge 0$ for $0 \le x \le 1$, we may conclude from the Intermediate Value Theorem that $f(x)$ has exactly one solution when $0 \le x \le 1$.
(b) $x^2 = \frac{-2 \pm \sqrt{4+8}}{2} > 0 \Rightarrow x^2 = \sqrt{3} - 1$ and $x \ge 0 \Rightarrow x \approx \sqrt{.7320508076} \approx .8555996772$

14. (a) $y = \frac{x}{x+1} \Rightarrow y' = \frac{1}{(x+1)^2} > 0$, for all x in the domain of $\frac{x}{x+1} \Rightarrow y = \frac{x}{x+1}$ is increasing in every interval in its domain.
(b) $y = x^3 + 2x \Rightarrow y' = 3x^2 + 2 > 0$ for all $x \Rightarrow$ the graph of $y = x^3 + 2x$ is always increasing and can never have a local maximum or minimum

15. Let $V(t)$ represent the volume of the water in the reservoir at time t, in minutes, let $V(0) = a_0$ be the initial amount and $V(1440) = a_0 + (1400)(43{,}560)(7.48)$ gallons be the amount of water contained in the reservoir after the rain, where $24 \text{ hr} = 1440 \text{ min}$. Assume that $V(t)$ is continuous on $[0, 1440]$ and differentiable on $(0, 1440)$. The Mean Value Theorem says that for some t_0 in $(0, 1440)$ we have $V'(t_0) = \frac{V(1440) - V(0)}{1440 - 0} = \frac{a_0 + (1400)(43{,}560)(7.48) - a_0}{1440} = \frac{456{,}160{,}320 \text{ gal}}{1440 \text{ min}}$ $= 316{,}778$ gal/min. Therefore at t_0 the reservoir's volume was increasing at a rate in excess of 225,000 gal/min.

16. Yes, all differentiable functions $g(x)$ having 3 as a derivative differ by only a constant. Consequently, the difference $3x - g(x)$ is a constant K because $g'(x) = 3 = \frac{d}{dx}(3x)$. Thus $g(x) = 3x + K$, the same form as $F(x)$.

17. No, $\frac{x}{x+1} = 1 + \frac{-1}{x+1} \Rightarrow \frac{x}{x+1}$ differs from $\frac{-1}{x+1}$ by the constant 1. Both functions have the same derivative $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2} = \frac{d}{dx}\left(\frac{-1}{x+1}\right)$.

18. $f'(x) = g'(x) = \frac{2x}{(x^2+1)^2} \Rightarrow f(x) - g(x) = C$ for some constant $C \Rightarrow$ the graphs differ by a vertical shift.

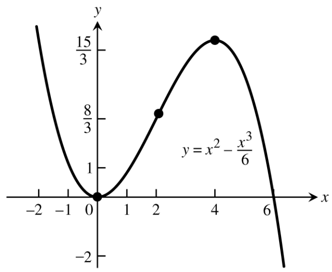
19. The global minimum value of $\frac{1}{2}$ occurs at $x = 2$.

20. (a) The function is increasing on the intervals $[-3, -2]$ and $[1, 2]$.
(b) The function is decreasing on the intervals $[-2, 0)$ and $(0, 1]$.
(c) The local maximum values occur only at $x = -2$, and at $x = 2$; local minimum values occur at $x = -3$ and at $x = 1$ provided f is continuous at $x = 0$.

21. (a) $t = 0, 6, 12$ (b) $t = 3, 9$ (c) $6 < t < 12$ (d) $0 < t < 6$, $12 < t < 14$

22. (a) $t = 4$ (b) at no time (c) $0 < t < 4$ (d) $4 < t < 8$

23.



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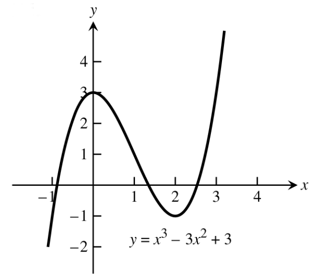

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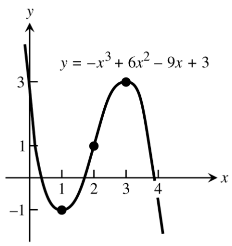

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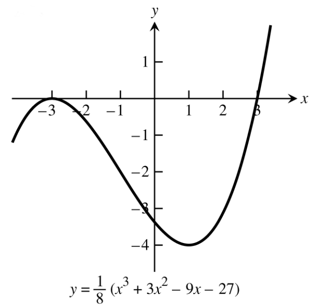

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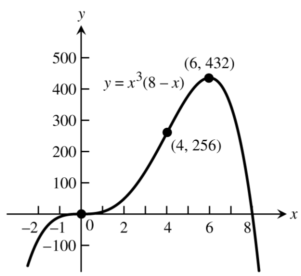

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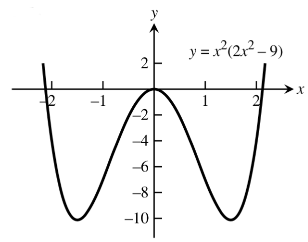

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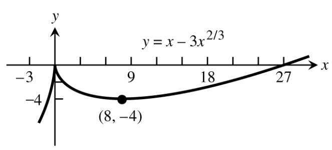
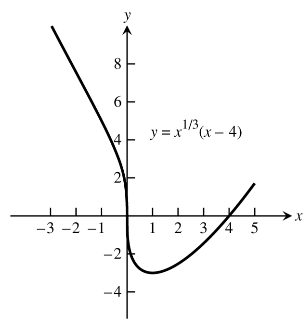
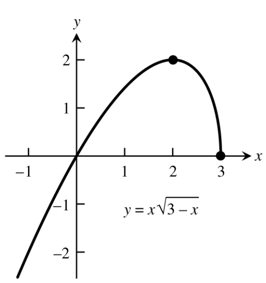


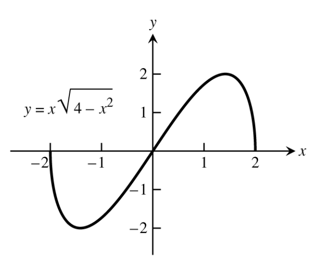
30.



31.

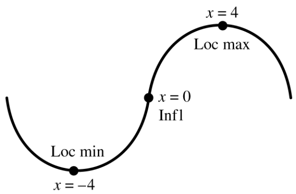


32.



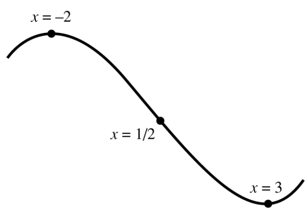
33. (a) $y' = 16 - x^2 \Rightarrow y' = ---\underset{-4}{|}+++\underset{4}{|}--- \Rightarrow$ the curve is rising on $(-4, 4)$, falling on $(-\infty, -4)$ and $(4, \infty)$ $\Rightarrow$ a local maximum at $x = 4$ and a local minimum at $x = -4$; $y'' = -2x \Rightarrow y'' = +++\underset{0}{|}--- \Rightarrow$ the curve is concave up on $(-\infty, 0)$, concave down on $(0, \infty) \Rightarrow$ a point of inflection at $x = 0$

(b)



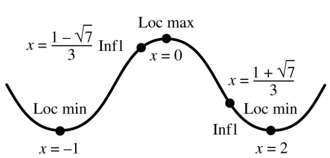
34. (a) $y' = x^2 - x - 6 = (x-3)(x+2) \Rightarrow y' = +++\underset{-2}{|}---\underset{3}{|}+++ \Rightarrow$ the curve is rising on $(-\infty, -2)$ and $(3, \infty)$, falling on $(-2, 3) \Rightarrow$ local maximum at $x = -2$ and a local minimum at $x = 3$; $y'' = 2x - 1$ $\Rightarrow y'' = ---\underset{1/2}{|}+++ \Rightarrow$ concave up on $\left(\frac{1}{2}, \infty\right)$, concave down on $\left(-\infty, \frac{1}{2}\right) \Rightarrow$ a point of inflection at $x = \frac{1}{2}$

(b)



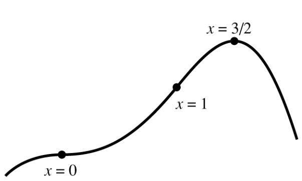
35. (a) $y' = 6x(x+1)(x-2) = 6x^3 - 6x^2 - 12x \Rightarrow y' = ---\underset{-1}{|}+++\underset{0}{|}---\underset{2}{|}+++ \Rightarrow$ the graph is rising on $(-1, 0)$ and $(2, \infty)$, falling on $(-\infty, -1)$ and $(0, 2) \Rightarrow$ a local maximum at $x = 0$, local minima at $x = -1$ and $x = 2$; $y'' = 18x^2 - 12x - 12 = 6\,(3x^2 - 2x - 2) = 6\left(x - \frac{1-\sqrt{7}}{3}\right)\left(x - \frac{1+\sqrt{7}}{3}\right) \Rightarrow$
$y'' = +++\underset{\frac{1-\sqrt{7}}{3}}{|}---\underset{\frac{1+\sqrt{7}}{3}}{|}+++ \Rightarrow$ the curve is concave up on $\left(-\infty, \frac{1-\sqrt{7}}{3}\right)$ and $\left(\frac{1+\sqrt{7}}{3}, \infty\right)$, concave down on $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right) \Rightarrow$ points of inflection at $x = \frac{1 \pm \sqrt{7}}{3}$

(b)



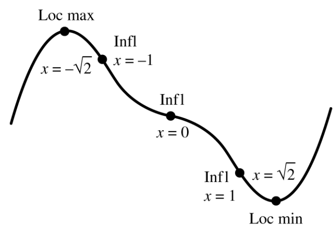
36. (a) $y' = x^2(6 - 4x) = 6x^2 - 4x^3 \Rightarrow y' = +++\underset{0}{|}+++\underset{3/2}{|}--- \Rightarrow$ the curve is rising on $\left(-\infty, \frac{3}{2}\right)$, falling on $\left(\frac{3}{2}, \infty\right)$ $\Rightarrow$ a local maximum at $x = \frac{3}{2}$; $y'' = 12x - 12x^2 = 12x(1-x) \Rightarrow y'' = ---\underset{0}{|}+++\underset{1}{|}--- \Rightarrow$ concave up on $(0, 1)$, concave down on $(-\infty, 0)$ and $(1, \infty) \Rightarrow$ points of inflection at $x = 0$ and $x = 1$

(b)



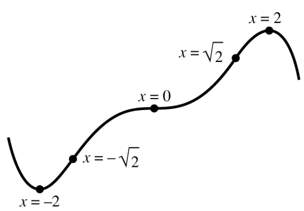
37. (a) $y' = x^4 - 2x^2 = x^2\,(x^2 - 2) \Rightarrow y' = +++ \underset{-\sqrt{2}}{|} --- \underset{0}{|} --- \underset{\sqrt{2}}{|} +++ \Rightarrow$ the curve is rising on $\left(-\infty, -\sqrt{2}\right)$ and $\left(\sqrt{2}, \infty\right)$, falling on $\left(-\sqrt{2}, \sqrt{2}\right) \Rightarrow$ a local maximum at $x = -\sqrt{2}$ and a local minimum at $x = \sqrt{2}$; $y'' = 4x^3 - 4x = 4x(x-1)(x+1) \Rightarrow y'' = --- \underset{-1}{|} +++ \underset{0}{|} --- \underset{1}{|} +++ \Rightarrow$ concave up on $(-1, 0)$ and $(1, \infty)$, concave down on $(-\infty, -1)$ and $(0, 1) \Rightarrow$ points of inflection at $x = 0$ and $x = \pm 1$

(b)

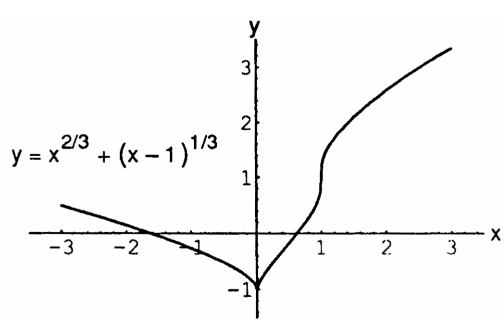


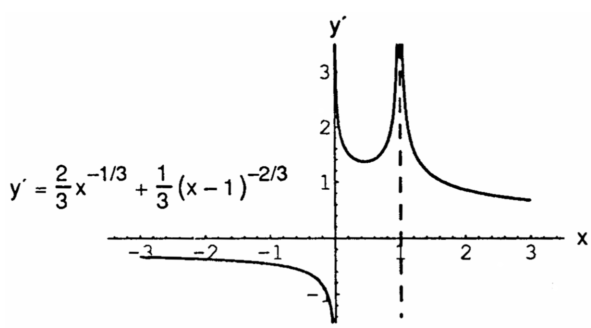
38. (a) $y' = 4x^2 - x^4 = x^2\,(4 - x^2) \Rightarrow y' = --- \underset{-2}{|} +++ \underset{0}{|} +++ \underset{2}{|} --- \Rightarrow$ the curve is rising on $(-2, 0)$ and $(0, 2)$, falling on $(-\infty, -2)$ and $(2, \infty) \Rightarrow$ a local maximum at $x = 2$, a local minimum at $x = -2$; $y'' = 8x - 4x^3 = 4x\,(2 - x^2) \Rightarrow y'' = +++ \underset{-\sqrt{2}}{|} --- \underset{0}{|} +++ \underset{\sqrt{2}}{|} --- \Rightarrow$ concave up on $\left(-\infty, -\sqrt{2}\right)$ and $\left(0, \sqrt{2}\right)$, concave down on $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, \infty\right) \Rightarrow$ points of inflection at $x = 0$ and $x = \pm\sqrt{2}$

(b)

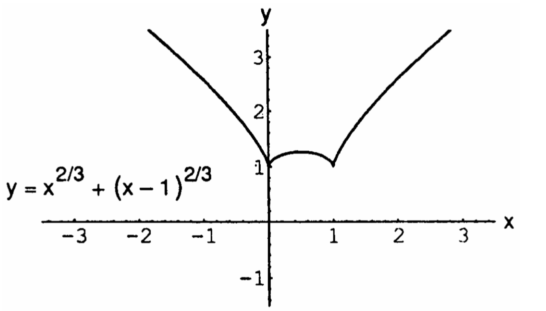


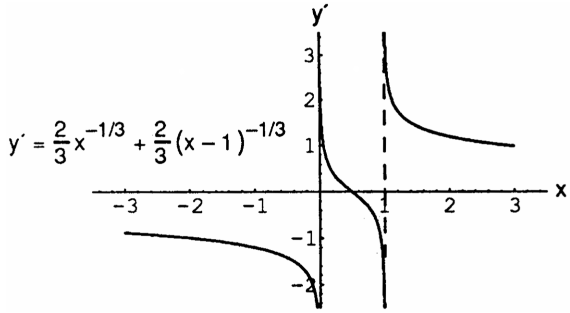
39. The values of the first derivative indicate that the curve is rising on $(0, \infty)$ and falling on $(-\infty, 0)$. The slope of the curve approaches $-\infty$ as $x \to 0^-$, and approaches ∞ as $x \to 0^+$ and $x \to 1$. The curve should therefore have a cusp and local minimum at $x = 0$, and a vertical tangent at $x = 1$.



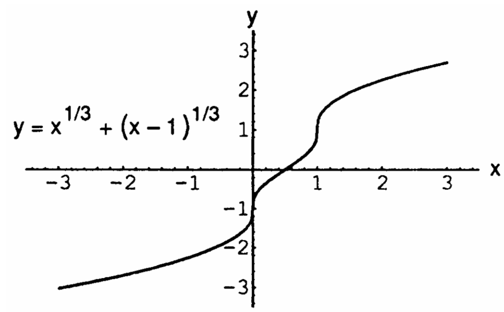


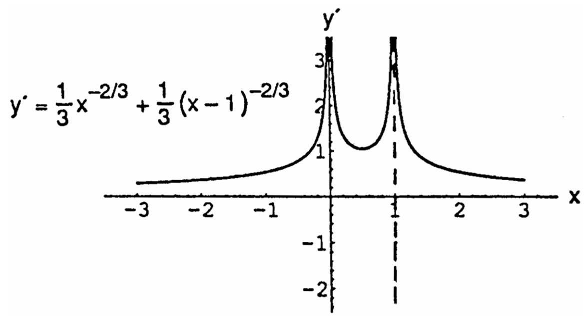
40. The values of the first derivative indicate that the curve is rising on $\left(0, \frac{1}{2}\right)$ and $(1, \infty)$, and falling on $(-\infty, 0)$ and $\left(\frac{1}{2}, 1\right)$. The derivative changes from positive to negative at $x = \frac{1}{2}$, indicating a local maximum there. The slope of the curve approaches $-\infty$ as $x \to 0^-$ and $x \to 1^-$, and approaches ∞ as $x \to 0^+$ and as $x \to 1^+$, indicating cusps and local minima at both $x = 0$ and $x = 1$.



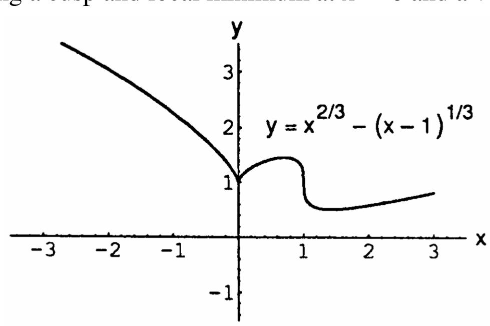


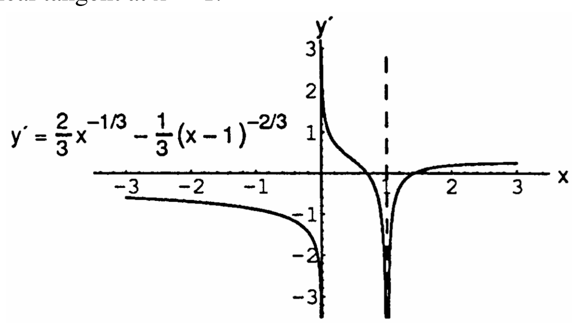
41. The values of the first derivative indicate that the curve is always rising. The slope of the curve approaches ∞ as $x \to 0$ and as $x \to 1$, indicating vertical tangents at both $x = 0$ and $x = 1$.



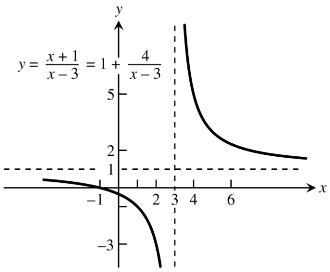


42. The graph of the first derivative indicates that the curve is rising on $\left(0, \frac{17-\sqrt{33}}{16}\right)$ and $\left(\frac{17+\sqrt{33}}{16}, \infty\right)$, falling on $(-\infty, 0)$ and $\left(\frac{17-\sqrt{33}}{16}, \frac{17+\sqrt{33}}{16}\right) \Rightarrow$ a local maximum at $x = \frac{17-\sqrt{33}}{16}$, a local minimum at $x = \frac{17+\sqrt{33}}{16}$. The derivative approaches $-\infty$ as $x \to 0^-$ and $x \to 1$, and approaches ∞ as $x \to 0^+$, indicating a cusp and local minimum at $x = 0$ and a vertical tangent at $x = 1$.

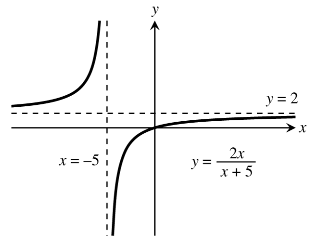




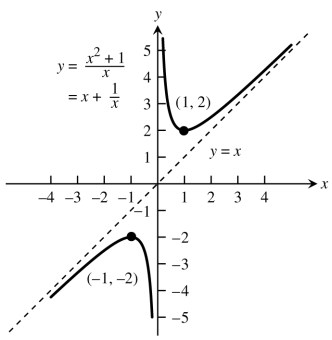
43. $y = \frac{x+1}{x-3} = 1 + \frac{4}{x-3}$



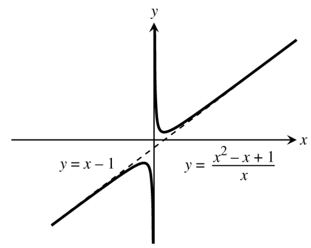
44. $y = \frac{2x}{x+5} = 2 - \frac{10}{x+5}$



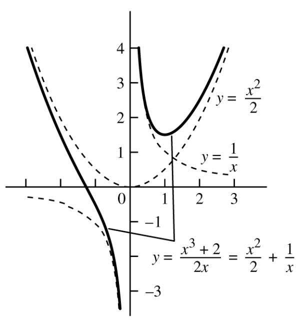
45. $y = \frac{x^2+1}{x} = x + \frac{1}{x}$



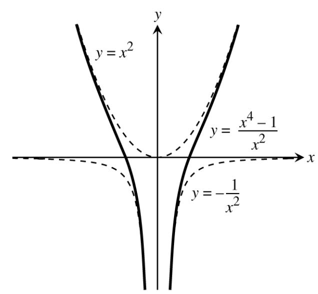
46. $y = \frac{x^2-x+1}{x} = x - 1 + \frac{1}{x}$



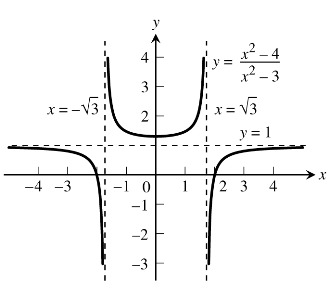
47. $y = \frac{x^3+2}{2x} = \frac{x^2}{2} + \frac{1}{x}$



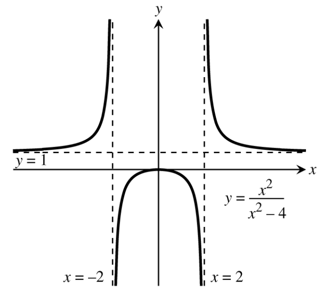
48. $y = \frac{x^4-1}{x^2} = x^2 - \frac{1}{x^2}$



49. $y = \frac{x^2-4}{x^2-3} = 1 - \frac{1}{x^2-3}$



50. $y = \frac{x^2}{x^2-4} = 1 + \frac{4}{x^2-4}$



51. (a) Maximize $f(x) = \sqrt{x} - \sqrt{36-x} = x^{1/2} - (36-x)^{1/2}$ where $0 \le x \le 36$
$\Rightarrow f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}(36-x)^{-1/2}(-1) = \frac{\sqrt{36-x}+\sqrt{x}}{2\sqrt{x}\sqrt{36-x}} \Rightarrow$ derivative fails to exist at 0 and 36; $f(0) = -6$, and $f(36) = 6 \Rightarrow$ the numbers are 0 and 36

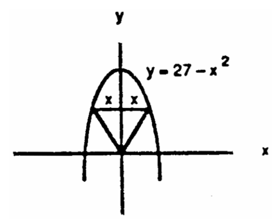
(b) Maximize $g(x) = \sqrt{x} + \sqrt{36 - x} = x^{1/2} + (36 - x)^{1/2}$ where $0 \le x \le 36$

$\Rightarrow g'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}(36 - x)^{-1/2}(-1) = \frac{\sqrt{36 - x} - \sqrt{x}}{2\sqrt{x}\sqrt{36 - x}} \Rightarrow$ critical points at 0, 18 and 36; $g(0) = 6$, $g(18) = 2\sqrt{18} = 6\sqrt{2}$ and $g(36) = 6 \Rightarrow$ the numbers are 18 and 18

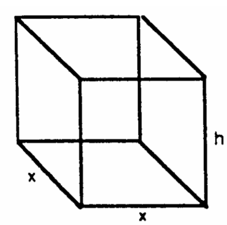
52. (a) Maximize $f(x) = \sqrt{x}(20 - x) = 20x^{1/2} - x^{3/2}$ where $0 \le x \le 20 \Rightarrow f'(x) = 10x^{-1/2} - \frac{3}{2}x^{1/2}$ $= \frac{20 - 3x}{2\sqrt{x}} = 0 \Rightarrow x = 0$ and $x = \frac{20}{3}$ are critical points; $f(0) = f(20) = 0$ and $f\left(\frac{20}{3}\right) = \sqrt{\frac{20}{3}}\left(20 - \frac{20}{3}\right)$ $= \frac{40\sqrt{20}}{3\sqrt{3}} \Rightarrow$ the numbers are $\frac{20}{3}$ and $\frac{40}{3}$.

(b) Maximize $g(x) = x + \sqrt{20 - x} = x + (20 - x)^{1/2}$ where $0 \le x \le 20 \Rightarrow g'(x) = \frac{2\sqrt{20 - x} - 1}{2\sqrt{20 - x}} = 0$ $\Rightarrow \sqrt{20 - x} = \frac{1}{2} \Rightarrow x = \frac{79}{4}$. The critical points are $x = \frac{79}{4}$ and $x = 20$. Since $g\left(\frac{79}{4}\right) = \frac{81}{4}$ and $g(20) = 20$, the numbers must be $\frac{79}{4}$ and $\frac{1}{4}$.

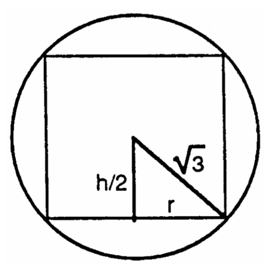
53. $A(x) = \frac{1}{2}(2x)(27 - x^2)$ for $0 \le x \le \sqrt{27}$ $\Rightarrow A'(x) = 3(3 + x)(3 - x)$ and $A''(x) = -6x$. The critical points are -3 and 3, but -3 is not in the domain. Since $A''(3) = -18 < 0$ and $A\left(\sqrt{27}\right) = 0$, the maximum occurs at $x = 3 \Rightarrow$ the largest area is $A(3) = 54$ sq units.



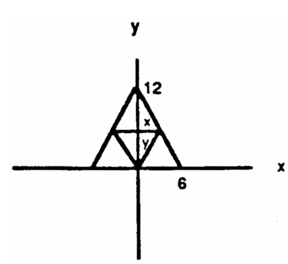
54. The volume is $V = x^2h = 32 \Rightarrow h = \frac{32}{x^2}$. The surface area is $S(x) = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}$, where $x > 0 \Rightarrow S'(x) = \frac{2(x - 4)(x^2 + 4x + 16)}{x^2}$ $\Rightarrow$ the critical points are 0 and 4, but 0 is not in the domain. Now $S''(4) = 2 + \frac{256}{4^3} > 0 \Rightarrow$ at $x = 4$ there is a minimum. The dimensions 4 ft by 4 ft by 2 ft minimize the surface area.



55. From the diagram we have $\left(\frac{h}{2}\right)^2 + r^2 = \left(\sqrt{3}\right)^2$ $\Rightarrow r^2 = \frac{12 - h^2}{4}$. The volume of the cylinder is $V = \pi r^2 h = \pi\left(\frac{12 - h^2}{4}\right)h = \frac{\pi}{4}(12h - h^3)$, where $0 \le h \le 2\sqrt{3}$. Then $V'(h) = \frac{3\pi}{4}(2 + h)(2 - h)$ $\Rightarrow$ the critical points are -2 and 2, but -2 is not in the domain. At $h = 2$ there is a maximum since $V''(2) = -3\pi < 0$. The dimensions of the largest cylinder are radius $= \sqrt{2}$ and height $= 2$.

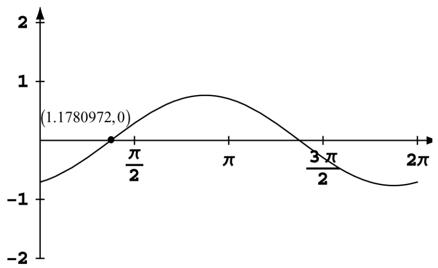


56. From the diagram we have x = radius and y = height $= 12 - 2x$ and $V(x) = \frac{1}{3}\pi x^2(12 - 2x)$, where $0 \le x \le 6 \Rightarrow V'(x) = 2\pi x(4 - x)$ and $V''(4) = -8\pi$. The critical points are 0 and 4; $V(0) = V(6) = 0 \Rightarrow x = 4$ gives the maximum. Thus the values of $r = 4$ and $h = 4$ yield the largest volume for the smaller cone.



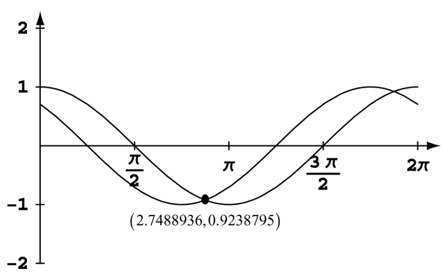
57. The profit $P = 2px + py = 2px + p\left(\frac{40 - 10x}{5 - x}\right)$, where p is the profit on grade B tires and $0 \le x \le 4$. Thus $P'(x) = \frac{2p}{(5 - x)^2}(x^2 - 10x + 20) \Rightarrow$ the critical points are $\left(5 - \sqrt{5}\right)$, 5, and $\left(5 + \sqrt{5}\right)$, but only $\left(5 - \sqrt{5}\right)$ is in the domain. Now $P'(x) > 0$ for $0 < x < \left(5 - \sqrt{5}\right)$ and $P'(x) < 0$ for $\left(5 - \sqrt{5}\right) < x < 4 \Rightarrow$ at $x = \left(5 - \sqrt{5}\right)$ there is a local maximum. Also $P(0) = 8p$, $P\left(5 - \sqrt{5}\right) = 4p\left(5 - \sqrt{5}\right) \approx 11p$, and $P(4) = 8p \Rightarrow$ at $x = \left(5 - \sqrt{5}\right)$ there is an absolute maximum. The maximum occurs when $x = \left(5 - \sqrt{5}\right)$ and $y = 2\left(5 - \sqrt{5}\right)$, the units are hundreds of tires, i.e., $x \approx 276$ tires and $y \approx 553$ tires.

58. (a) The distance between the particles is $|f(t)|$ where $f(t) = -\cos t + \cos\left(t + \frac{\pi}{4}\right)$. Then, $f'(t) = \sin t - \sin\left(t + \frac{\pi}{4}\right)$. Solving $f'(t) = 0$ graphically, we obtain $t \approx 1.178$, $t \approx 4.320$, and so on.



Alternatively, $f'(t) = 0$ may be solved analytically as follows. $f'(t) = \sin\left[\left(t + \frac{\pi}{8}\right) - \frac{\pi}{8}\right] - \sin\left[\left(t + \frac{\pi}{8}\right) + \frac{\pi}{8}\right]$
$= \left[\sin\left(t + \frac{\pi}{8}\right)\cos\frac{\pi}{8} - \cos\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8}\right] - \left[\sin\left(t + \frac{\pi}{8}\right)\cos\frac{\pi}{8} + \cos\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8}\right] = -2\sin\frac{\pi}{8}\cos\left(t + \frac{\pi}{8}\right)$
so the critical points occur when $\cos\left(t + \frac{\pi}{8}\right) = 0$, or $t = \frac{3\pi}{8} + k\pi$. At each of these values, $f(t) = \pm\cos\frac{3\pi}{8}$ $\approx \pm 0.765$ units, so the maximum distance between the particles is 0.765 units.

(b) Solving $\cos t = \cos\left(t + \frac{\pi}{4}\right)$ graphically, we obtain $t \approx 2.749$, $t \approx 5.890$, and so on.



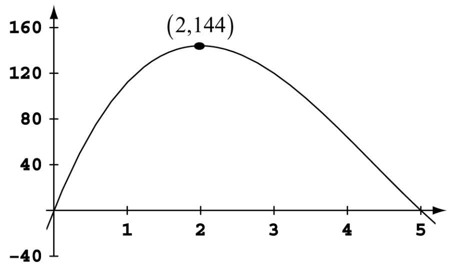
Alternatively, this problem can be solved analytically as follows.

$$\cos t = \cos\left(t + \frac{\pi}{4}\right)$$
$$\cos\left[\left(t + \frac{\pi}{8}\right) - \frac{\pi}{8}\right] = \cos\left[\left(t + \frac{\pi}{8}\right) + \frac{\pi}{8}\right]$$
$$\cos\left(t + \frac{\pi}{8}\right)\cos\frac{\pi}{8} + \sin\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8} = \cos\left(t + \frac{\pi}{8}\right)\cos\frac{\pi}{8} - \sin\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8}$$
$$2\sin\left(t + \frac{\pi}{8}\right)\sin\frac{\pi}{8} = 0$$
$$\sin\left(t + \frac{\pi}{8}\right) = 0$$
$$t = \frac{7\pi}{8} + k\pi$$

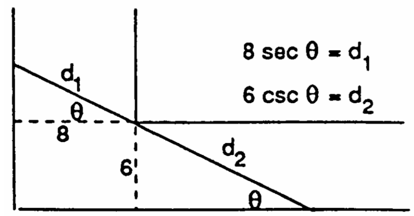
The particles collide when $t = \frac{7\pi}{8} \approx 2.749$. (plus multiples of π if they keep going.)

59. The dimensions will be x in. by $10 - 2x$ in. by $16 - 2x$ in., so $V(x) = x(10 - 2x)(16 - 2x) = 4x^3 - 52x^2 + 160x$ for $0 < x < 5$. Then $V'(x) = 12x^2 - 104x + 160 = 4(x - 2)(3x - 20)$, so the critical point in the correct domain is $x = 2$. This critical point corresponds to the maximum possible volume because $V'(x) > 0$ for $0 < x < 2$ and $V'(x) < 0$ for $2 < x < 5$. The box of largest volume has a height of 2 in. and a base measuring 6 in. by 12 in., and its volume is 144 in.^{3}

Graphical support:



60. The length of the ladder is $d_1 + d_2 = 8 \sec\theta + 6 \csc\theta$. We wish to maximize $I(\theta) = 8 \sec\theta + 6 \csc\theta \Rightarrow I'(\theta) = 8 \sec\theta \tan\theta - 6 \csc\theta \cot\theta$. Then $I'(\theta) = 0 \Rightarrow 8 \sin^3\theta - 6 \cos^3\theta = 0 \Rightarrow \tan\theta = \frac{\sqrt[3]{6}}{2} \Rightarrow$ $d_1 = 4\sqrt{4 + \sqrt[3]{36}}$ and $d_2 = \sqrt[3]{36}\sqrt{4 + \sqrt[3]{36}}$ $\Rightarrow$ the length of the ladder is about $\left(4 + \sqrt[3]{36}\right)\sqrt{4 + \sqrt[3]{36}} = \left(4 + \sqrt[3]{36}\right)^{3/2} \approx 19.7$ ft.



61. $g(x) = 3x - x^3 + 4 \Rightarrow g(2) = 2 > 0$ and $g(3) = -14 < 0 \Rightarrow g(x) = 0$ in the interval $[2, 3]$ by the Intermediate Value Theorem. Then $g'(x) = 3 - 3x^2 \Rightarrow x_{n+1} = x_n - \frac{3x_n - x_n^3 + 4}{3 - 3x_n^2}$; $x_0 = 2 \Rightarrow x_1 = 2.\overline{22} \Rightarrow x_2 = 2.196215$, and so forth to $x_5 = 2.195823345$.

62. $g(x) = x^4 - x^3 - 75 \Rightarrow g(3) = -21 < 0$ and $g(4) = 117 > 0 \Rightarrow g(x) = 0$ in the interval $[3, 4]$ by the Intermediate Value Theorem. Then $g'(x) = 4x^3 - 3x^2 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 - x_n^3 - 75}{4x_n^3 - 3x_n^2}$; $x_0 = 3 \Rightarrow x_1 = 3.259259$ $\Rightarrow x_2 = 3.229050$, and so forth to $x_5 = 3.22857729$.

63. $\int (x^3 + 5x - 7)\,dx = \frac{x^4}{4} + \frac{5x^2}{2} - 7x + C$

64. $\int \left(8t^3 - \frac{t^2}{2} + t\right) dt = \frac{8t^4}{4} - \frac{t^3}{6} + \frac{t^2}{2} + C = 2t^4 - \frac{t^3}{6} + \frac{t^2}{2} + C$

65. $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt = \int \left(3t^{1/2} + 4t^{-2}\right) dt = \frac{3t^{3/2}}{\left(\frac{3}{2}\right)} + \frac{4t^{-1}}{-1} + C = 2t^{3/2} - \frac{4}{t} + C$

66. $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt = \int \left(\frac{1}{2}t^{-1/2} - 3t^{-4}\right) dt = \frac{1}{2}\left(\frac{t^{1/2}}{\frac{1}{2}}\right) - \frac{3t^{-3}}{(-3)} + C = \sqrt{t} + \frac{1}{t^3} + C$

67. Let $u = r + 5 \Rightarrow du = dr$
$\int \frac{dr}{(r+5)^2} = \int \frac{du}{u^2} = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C = -u^{-1} + C = -\frac{1}{(r+5)} + C$

68. Let $u = r - \sqrt{2} \Rightarrow du = dr$
$\int \frac{6\,dr}{\left(r - \sqrt{2}\right)^3} = 6\int \frac{dr}{\left(r - \sqrt{2}\right)^3} = 6\int \frac{du}{u^3} = 6\int u^{-3}\,du = 6\left(\frac{u^{-2}}{-2}\right) + C = -3u^{-2} + C = -\frac{3}{\left(r - \sqrt{2}\right)^2} + C$

69. Let $u = \theta^2 + 1 \Rightarrow du = 2\theta\, d\theta \Rightarrow \frac{1}{2}\, du = \theta\, d\theta$

$$\int 3\theta\sqrt{\theta^2+1}\, d\theta = \int \sqrt{u}\left(\tfrac{3}{2}\, du\right) = \tfrac{3}{2}\int u^{1/2}\, du = \tfrac{3}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + C = u^{3/2} + C = \left(\theta^2+1\right)^{3/2} + C$$

70. Let $u = 7 + \theta^2 \Rightarrow du = 2\theta\, d\theta \Rightarrow \frac{1}{2}\, du = \theta\, d\theta$

$$\int \frac{\theta}{\sqrt{7+\theta^2}}\, d\theta = \int \frac{1}{\sqrt{u}}\left(\tfrac{1}{2}\, du\right) = \tfrac{1}{2}\int u^{-1/2}\, du = \tfrac{1}{2}\left(\frac{u^{1/2}}{\frac{1}{2}}\right) + C = u^{1/2} + C = \sqrt{7+\theta^2} + C$$

71. Let $u = 1 + x^4 \Rightarrow du = 4x^3\, dx \Rightarrow \frac{1}{4}\, du = x^3\, dx$

$$\int x^3\left(1+x^4\right)^{-1/4} dx = \int u^{-1/4}\left(\tfrac{1}{4}\, du\right) = \tfrac{1}{4}\int u^{-1/4}\, du = \tfrac{1}{4}\left(\frac{u^{3/4}}{\frac{3}{4}}\right) + C = \tfrac{1}{3}u^{3/4} + C = \tfrac{1}{3}\left(1+x^4\right)^{3/4} + C$$

72. Let $u = 2 - x \Rightarrow du = -\, dx \Rightarrow -\, du = dx$

$$\int (2-x)^{3/5}\, dx = \int u^{3/5}(-\, du) = -\int u^{3/5}\, du = -\frac{u^{8/5}}{\left(\frac{8}{5}\right)} + C = -\tfrac{5}{8}u^{8/5} + C = -\tfrac{5}{8}(2-x)^{8/5} + C$$

73. Let $u = \frac{s}{10} \Rightarrow du = \frac{1}{10}\, ds \Rightarrow 10\, du = ds$

$$\int \sec^2 \tfrac{s}{10}\, ds = \int \left(\sec^2 u\right)(10\, du) = 10\int \sec^2 u\, du = 10\tan u + C = 10\tan \tfrac{s}{10} + C$$

74. Let $u = \pi s \Rightarrow du = \pi\, ds \Rightarrow \frac{1}{\pi}\, du = ds$

$$\int \csc^2 \pi s\, ds = \int \left(\csc^2 u\right)\left(\tfrac{1}{\pi}\, du\right) = \tfrac{1}{\pi}\int \csc^2 u\, du = -\tfrac{1}{\pi}\cot u + C = -\tfrac{1}{\pi}\cot \pi s + C$$

75. Let $u = \sqrt{2}\,\theta \Rightarrow du = \sqrt{2}\, d\theta \Rightarrow \frac{1}{\sqrt{2}}\, du = d\theta$

$$\int \csc \sqrt{2}\theta \cot \sqrt{2}\theta\, d\theta = \int (\csc u \cot u)\left(\frac{1}{\sqrt{2}}\, du\right) = \frac{1}{\sqrt{2}}(-\csc u) + C = -\frac{1}{\sqrt{2}}\csc \sqrt{2}\theta + C$$

76. Let $u = \frac{\theta}{3} \Rightarrow du = \frac{1}{3}\, d\theta \Rightarrow 3\, du = d\theta$

$$\int \sec \tfrac{\theta}{3}\tan \tfrac{\theta}{3}\, d\theta = \int (\sec u \tan u)(3\, du) = 3\sec u + C = 3\sec \tfrac{\theta}{3} + C$$

77. Let $u = \frac{x}{4} \Rightarrow du = \frac{1}{4}\, dx \Rightarrow 4\, du = dx$

$$\int \sin^2 \tfrac{x}{4}\, dx = \int \left(\sin^2 u\right)(4\, du) = \int 4\left(\tfrac{1-\cos 2u}{2}\right) du = 2\int (1-\cos 2u)\, du = 2\left(u - \tfrac{\sin 2u}{2}\right) + C$$
$$= 2u - \sin 2u + C = 2\left(\tfrac{x}{4}\right) - \sin 2\left(\tfrac{x}{4}\right) + C = \tfrac{x}{2} - \sin \tfrac{x}{2} + C$$

78. Let $u = \frac{x}{2} \Rightarrow du = \frac{1}{2}\, dx \Rightarrow 2\, du = dx$

$$\int \cos^2 \tfrac{x}{2}\, dx = \int \left(\cos^2 u\right)(2\, du) = \int 2\left(\tfrac{1+\cos 2u}{2}\right) du = \int (1+\cos 2u)\, du = u + \tfrac{\sin 2u}{2} + C = \tfrac{x}{2} + \tfrac{1}{2}\sin x + C$$

79. $y = \int \frac{x^2+1}{x^2}\, dx = \int \left(1 + x^{-2}\right) dx = x - x^{-1} + C = x - \frac{1}{x} + C$; $y = -1$ when $x = 1 \Rightarrow 1 - \frac{1}{1} + C = -1$
$\Rightarrow C = -1 \Rightarrow y = x - \frac{1}{x} - 1$

80. $y = \int \left(x + \frac{1}{x}\right)^2 dx = \int \left(x^2 + 2 + \frac{1}{x^2}\right) dx = \int \left(x^2 + 2 + x^{-2}\right) dx = \frac{x^3}{3} + 2x - x^{-1} + C = \frac{x^3}{3} + 2x - \frac{1}{x} + C$;
$y = 1$ when $x = 1 \Rightarrow \frac{1}{3} + 2 - \frac{1}{1} + C = 1 \Rightarrow C = -\frac{1}{3} \Rightarrow y = \frac{x^3}{3} + 2x - \frac{1}{x} - \frac{1}{3}$

81. $\frac{dr}{dt} = \int \left(15\sqrt{t} + \frac{3}{\sqrt{t}}\right) dt = \int \left(15t^{1/2} + 3t^{-1/2}\right) dt = 10t^{3/2} + 6t^{1/2} + C$; $\frac{dr}{dt} = 8$ when $t = 1$
$\Rightarrow 10(1)^{3/2} + 6(1)^{1/2} + C = 8 \Rightarrow C = -8$. Thus $\frac{dr}{dt} = 10t^{3/2} + 6t^{1/2} - 8 \Rightarrow r = \int \left(10t^{3/2} + 6t^{1/2} - 8\right) dt$

$= 4t^{5/2} + 4t^{3/2} - 8t + C$; $r = 0$ when $t = 1 \Rightarrow 4(1)^{5/2} + 4(1)^{3/2} - 8(1) + C_1 = 0 \Rightarrow C_1 = 0$. Therefore, $r = 4t^{5/2} + 4t^{3/2} - 8t$

82. $\frac{d^2r}{dt^2} = \int -\cos t\, dt = -\sin t + C$; $r'' = 0$ when $t = 0 \Rightarrow -\sin 0 + C = 0 \Rightarrow C = 0$. Thus, $\frac{d^2r}{dt^2} = -\sin t$ $\Rightarrow \frac{dr}{dt} = \int -\sin t\, dt = \cos t + C_1$; $r' = 0$ when $t = 0 \Rightarrow 1 + C_1 = 0 \Rightarrow C_1 = -1$. Then $\frac{dr}{dt} = \cos t - 1$ $\Rightarrow r = \int (\cos t - 1)\, dt = \sin t - t + C_2$; $r = -1$ when $t = 0 \Rightarrow 0 - 0 + C_2 = -1 \Rightarrow C_2 = -1$. Therefore, $r = \sin t - t - 1$

CHAPTER 4 ADDITIONAL AND ADVANCED EXERCISES

1. If M and m are the maximum and minimum values, respectively, then $m \le f(x) \le M$ for all $x \in I$. If $m = M$ then f is constant on I.

2. No, the function $f(x) = \begin{cases} 3x + 6, & -2 \le x < 0 \\ 9 - x^2, & 0 \le x \le 2 \end{cases}$ has an absolute minimum value of 0 at $x = -2$ and an absolute maximum value of 9 at $x = 0$, but it is discontinuous at $x = 0$.

3. On an open interval the extreme values of a continuous function (if any) must occur at an interior critical point. On a half-open interval the extreme values of a continuous function may be at a critical point or at the closed endpoint. Extreme values occur only where $f' = 0$, f' does not exist, or at the endpoints of the interval. Thus the extreme points will not be at the ends of an open interval.

4. The pattern $f' = +++ \underset{1}{|} ---- \underset{2}{|} ---- \underset{3}{|} ++++ \underset{4}{|} +++$ indicates a local maximum at $x = 1$ and a local minimum at $x = 3$.

5. (a) If $y' = 6(x+1)(x-2)^2$, then $y' < 0$ for $x < -1$ and $y' > 0$ for $x > -1$. The sign pattern is $f' = --- \underset{-1}{|} +++ \underset{2}{|} +++ \Rightarrow$ f has a local minimum at $x = -1$. Also $y'' = 6(x-2)^2 + 12(x+1)(x-2)$ $= 6(x-2)(3x) \Rightarrow y'' > 0$ for $x < 0$ or $x > 2$, while $y'' < 0$ for $0 < x < 2$. Therefore f has points of inflection at $x = 0$ and $x = 2$. There is no local maximum.
   (b) If $y' = 6x(x+1)(x-2)$, then $y' < 0$ for $x < -1$ and $0 < x < 2$; $y' > 0$ for $-1 < x < 0$ and $x > 2$. The sign sign pattern is $y' = --- \underset{-1}{|} +++ \underset{0}{|} --- \underset{2}{|} +++$. Therefore f has a local maximum at $x = 0$ and local minima at $x = -1$ and $x = 2$. Also, $y'' = 18\left[x - \left(\frac{1-\sqrt{7}}{3}\right)\right]\left[x - \left(\frac{1+\sqrt{7}}{3}\right)\right]$, so $y'' < 0$ for $\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$ and $y'' > 0$ for all other x $\Rightarrow$ f has points of inflection at $x = \frac{1 \pm \sqrt{7}}{3}$.

6. The Mean Value Theorem indicates that $\frac{f(6) - f(0)}{6 - 0} = f'(c) \le 2$ for some c in $(0, 6)$. Then $f(6) - f(0) \le 12$ indicates the most that f can increase is 12.

7. If f is continuous on $[a, c)$ and $f'(x) \le 0$ on $[a, c)$, then by the Mean Value Theorem for all $x \in [a, c)$ we have $\frac{f(c) - f(x)}{c - x} \le 0 \Rightarrow f(c) - f(x) \le 0 \Rightarrow f(x) \ge f(c)$. Also if f is continuous on $(c, b]$ and $f'(x) \ge 0$ on $(c, b]$, then for all $x \in (c, b]$ we have $\frac{f(x) - f(c)}{x - c} \ge 0 \Rightarrow f(x) - f(c) \ge 0 \Rightarrow f(x) \ge f(c)$. Therefore $f(x) \ge f(c)$ for all $x \in [a, b]$.

8. (a) For all x, $-(x+1)^2 \le 0 \le (x-1)^2 \Rightarrow -(1 + x^2) \le 2x \le (1 + x^2) \Rightarrow -\frac{1}{2} \le \frac{x}{1+x^2} \le \frac{1}{2}$.
   (b) There exists $c \in (a, b)$ such that $\frac{c}{1+c^2} = \frac{f(b) - f(a)}{b - a} \Rightarrow \left|\frac{f(b) - f(a)}{b - a}\right| = \left|\frac{c}{1+c^2}\right| \le \frac{1}{2}$, from part (a) $\Rightarrow |f(b) - f(a)| \le \frac{1}{2}|b - a|$.

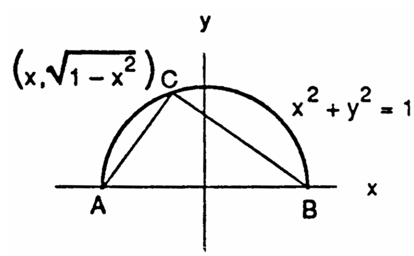
9. No. Corollary 1 requires that $f'(x)=0$ for <u>all</u> x in some interval I, not $f'(x)=0$ at a single point in I.

10. (a) $h(x)=f(x)g(x) \Rightarrow h'(x)=f'(x)g(x)+f(x)g'(x)$ which changes signs at $x=a$ since $f'(x), g'(x)>0$ when $x<a$, $f'(x), g'(x)<0$ when $x>a$ and $f(x), g(x)>0$ for all x. Therefore h(x) does have a local maximum at $x=a$.
(b) No, let $f(x)=g(x)=x^3$ which have points of inflection at $x=0$, but $h(x)=x^6$ has no point of inflection (it has a local minimum at $x=0$).

11. From (ii), $f(-1)=\frac{-1+a}{b-c+2}=0 \Rightarrow a=1$; from (iii), either $1=\lim_{x\to\infty} f(x)$ or $1=\lim_{x\to-\infty} f(x)$. In either case, $\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x+1}{bx^2+cx+2}=\lim_{x\to\pm\infty}\frac{1+\frac{1}{x}}{bx+c+\frac{2}{x}}=1 \Rightarrow b=0$ and $c=1$. For if $b=1$, then $\lim_{x\to\pm\infty}\frac{1+\frac{1}{x}}{x+c+\frac{2}{x}}=0$ and if $c=0$, then $\lim_{x\to\pm\infty}\frac{1+\frac{1}{x}}{bx+\frac{2}{x}}=\lim_{x\to\pm\infty}\frac{1+\frac{1}{x}}{\frac{2}{x}}=\pm\infty$. Thus $a=1$, $b=0$, and $c=1$.

12. $\frac{dy}{dx}=3x^2+2kx+3=0 \Rightarrow x=\frac{-2k\pm\sqrt{4k^2-36}}{6} \Rightarrow$ x has only one value when $4k^2-36=0 \Rightarrow k^2=9$ or $k=\pm3$.

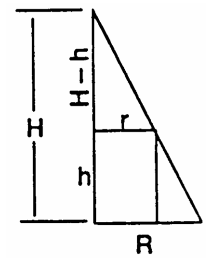
13. The area of the ΔABC is $A(x)=\frac{1}{2}(2)\sqrt{1-x^2}=(1-x^2)^{1/2}$, where $0\le x\le 1$. Thus $A'(x)=\frac{-x}{\sqrt{1-x^2}} \Rightarrow 0$ and ±1 are critical points. Also $A(\pm1)=0$ so $A(0)=1$ is the maximum. When $x=0$ the ΔABC is isosceles since $AC=BC=\sqrt{2}$.



14. $\lim_{h\to0}\frac{f'(c+h)-f'(c)}{h}=f''(c) \Rightarrow$ for $\epsilon=\frac{1}{2}|f''(c)|>0$ there exists a $\delta>0$ such that $0<|h|<\delta$
$\Rightarrow \left|\frac{f'(c+h)-f'(c)}{h}-f''(c)\right|<\frac{1}{2}|f''(c)|$. Then $f'(c)=0 \Rightarrow -\frac{1}{2}|f''(c)|<\frac{f'(c+h)}{h}-f''(c)<\frac{1}{2}|f''(c)|$
$\Rightarrow f''(c)-\frac{1}{2}|f''(c)|<\frac{f'(c+h)}{h}<f''(c)+\frac{1}{2}|f''(c)|$. If $f''(c)<0$, then $|f''(c)|=-f''(c)$
$\Rightarrow \frac{3}{2}f''(c)<\frac{f'(c+h)}{h}<\frac{1}{2}f''(c)<0$; likewise if $f''(c)>0$, then $0<\frac{1}{2}f''(c)<\frac{f'(c+h)}{h}<\frac{3}{2}f''(c)$.
(a) If $f''(c)<0$, then $-\delta<h<0 \Rightarrow f'(c+h)>0$ and $0<h<\delta \Rightarrow f'(c+h)<0$. Therefore, f(c) is a local maximum.
(b) If $f''(c)>0$, then $-\delta<h<0 \Rightarrow f'(c+h)<0$ and $0<h<\delta \Rightarrow f'(c+h)>0$. Therefore, f(c) is a local minimum.

15. The time it would take the water to hit the ground from height y is $\sqrt{\frac{2y}{g}}$, where g is the acceleration of gravity. The product of time and exit velocity (rate) yields the distance the water travels:
$D(y)=\sqrt{\frac{2y}{g}}\sqrt{64(h-y)}=8\sqrt{\frac{2}{g}}(hy-y^2)^{1/2}, 0\le y\le h \Rightarrow D'(y)=-4\sqrt{\frac{2}{g}}(hy-y^2)^{-1/2}(h-2y) \Rightarrow 0, \frac{h}{2}$ and h are critical points. Now $D(0)=0$, $D\left(\frac{h}{2}\right)=8\sqrt{\frac{2}{g}}\left(h\left(\frac{h}{2}\right)-\left(\frac{h}{2}\right)^2\right)^{1/2}=4h\sqrt{\frac{2}{g}}$ and $D(h)=0 \Rightarrow$ the best place to drill the hole is at $y=\frac{h}{2}$.

16. From the figure in the text, $\tan(\beta+\theta)=\frac{b+a}{h}$; $\tan(\beta+\theta)=\frac{\tan\beta+\tan\theta}{1-\tan\beta\tan\theta}$; and $\tan\theta=\frac{a}{h}$. These equations give $\frac{b+a}{h}=\frac{\tan\beta+\frac{a}{h}}{1-\frac{a}{h}\tan\beta}=\frac{h\tan\beta+a}{h-a\tan\beta}$. Solving for $\tan\beta$ gives $\tan\beta=\frac{bh}{h^2+a(b+a)}$ or $(h^2-a(b+a))\tan\beta=bh$. Differentiating both sides with respect to h gives
$2h\tan\beta+(h^2+a(b+a))\sec^2\beta\frac{d\beta}{dh}=b$. Then $\frac{d\beta}{dh}=0 \Rightarrow 2h\tan\beta=b \Rightarrow 2h\left(\frac{bh}{h^2+a(b+a)}\right)=b$
$\Rightarrow 2bh^2=bh^2+ab(b+a) \Rightarrow h^2=a(b+a) \Rightarrow h=\sqrt{a(a+b)}$.

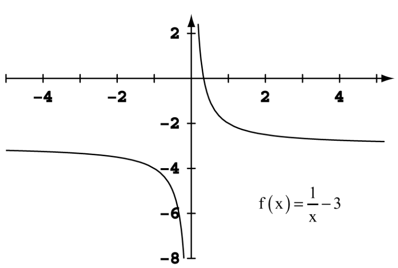


17. The surface area of the cylinder is $S = 2\pi r^2 + 2\pi rh$. From the diagram we have $\frac{r}{R} = \frac{H-h}{H} \Rightarrow h = \frac{RH - rH}{R}$ and $S(r) = 2\pi r(r+h) = 2\pi r\left(r + H - r\frac{H}{R}\right)$
$= 2\pi\left(1 - \frac{H}{R}\right)r^2 + 2\pi Hr$, where $0 \le r \le R$.
Case 1: $H < R \Rightarrow S(r)$ is a quadratic equation containing the origin and concave upward $\Rightarrow S(r)$ is maximum at $r = R$.
Case 2: $H = R \Rightarrow S(r)$ is a linear equation containing the origin with a positive slope $\Rightarrow S(r)$ is maximum at $r = R$.
Case 3: $H > R \Rightarrow S(r)$ is a quadratic equation containing the origin and concave downward. Then $\frac{dS}{dr} = 4\pi\left(1 - \frac{H}{R}\right)r + 2\pi H$ and $\frac{dS}{dr} = 0 \Rightarrow 4\pi\left(1 - \frac{H}{R}\right)r + 2\pi H = 0 \Rightarrow r = \frac{RH}{2(H-R)}$. For simplification we let $r^* = \frac{RH}{2(H-R)}$.
(a) If $R < H < 2R$, then $0 > H - 2R \Rightarrow H > 2(H-R) \Rightarrow r^* = \frac{RH}{2(H-R)} > R$. Therefore, the maximum occurs at the right endpoint R of the interval $0 \le r \le R$ because S(r) is an increasing function of r.
(b) If $H = 2R$, then $r^* = \frac{2R^2}{2R} = R \Rightarrow S(r)$ is maximum at $r = R$.
(c) If $H > 2R$, then $2R + H < 2H \Rightarrow H < 2(H-R) \Rightarrow \frac{H}{2(H-R)} < 1 \Rightarrow \frac{RH}{2(H-R)} < R \Rightarrow r^* < R$. Therefore, S(r) is a maximum at $r = r^* = \frac{RH}{2(H-R)}$.
Conclusion: If $H \in (0, 2R]$, then the maximum surface area is at $r = R$. If $H \in (2R, \infty)$, then the maximum is at $r = r^* = \frac{RH}{2(H-R)}$.

18. $f(x) = mx - 1 + \frac{1}{x} \Rightarrow f'(x) = m - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3} > 0$ when $x > 0$. Then $f'(x) = 0 \Rightarrow x = \frac{1}{\sqrt{m}}$ yields a minimum. If $f\left(\frac{1}{\sqrt{m}}\right) \ge 0$, then $\sqrt{m} - 1 + \sqrt{m} = 2\sqrt{m} - 1 \ge 0 \Rightarrow m \ge \frac{1}{4}$. Thus the smallest acceptable value for m is $\frac{1}{4}$.

19. (a) The profit function is $P(x) = (c - ex)x - (a + bx) = -ex^2 + (c - b)x - a$. $P'(x) = -2ex + c - b = 0 \Rightarrow x = \frac{c-b}{2e}$. $P''(x) = -2e < 0$ if $e > 0$ so that the profit function is maximized at $x = \frac{c-b}{2e}$.
(b) The price therefore that corresponds to a production level yeilding a maximum profit is $p\Big|_{x=\frac{c-b}{2e}} = c - e\left(\frac{c-b}{2e}\right) = \frac{c+b}{2}$ dollars.
(c) The weekly profit at this production level is $P(x) = -e\left(\frac{c-b}{2e}\right)^2 + (c-b)\left(\frac{c-b}{2e}\right) - a = \frac{(c-b)^2}{4e} - a$.
(d) The tax increases cost to the new profit function is $F(x) = (c - ex)x - (a + bx + tx) = -ex^2 + (c - b - t)x - a$. Now $F'(x) = -2ex + c - b - t = 0$ when $x = \frac{t+b-c}{-2e} = \frac{c-b-t}{2e}$. Since $F''(x) = -2e < 0$ if $e > 0$, F is maximized when $x = \frac{c-b-t}{2e}$ units per week. Thus the price per unit is $p = c - e\left(\frac{c-b-t}{2e}\right) = \frac{c+b+t}{2}$ dollars. Thus, such a tax increases the cost per unit by $\frac{c+b+t}{2} - \frac{c+b}{2} = \frac{t}{2}$ dollars if units are priced to maximize profit.

20. (a)



The x-intercept occurs when $\frac{1}{x} - 3 = 0 \Rightarrow \frac{1}{x} = 3 \Rightarrow x = \frac{1}{3}$.

(b) By Newton's method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Here $f'(x_n) = -x_n^{-2} = \frac{-1}{x_n^2}$. So $x_{n+1} = x_n - \frac{\frac{1}{x_n}-3}{\frac{-1}{x_n^2}} = x_n + \left(\frac{1}{x_n} - 3\right)x_n^2$
$= x_n + x_n - 3x_n^2 = 2x_n - 3x_n^2 = x_n(2 - 3x_n)$.

21. $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^q - a}{qx_0^{q-1}} = \frac{qx_0^q - x_0^q + a}{qx_0^{q-1}} = \frac{x_0^q(q-1)+a}{qx_0^{q-1}} = x_0\left(\frac{q-1}{q}\right) + \frac{a}{x_0^{q-1}}\left(\frac{1}{q}\right)$ so that x_1 is a weighted average of x_0 and $\frac{a}{x_0^{q-1}}$ with weights $m_0 = \frac{q-1}{q}$ and $m_1 = \frac{1}{q}$.
In the case where $x_0 = \frac{a}{x_0^{q-1}}$ we have $x_0^q = a$ and $x_1 = \frac{a}{x_0^{q-1}}\left(\frac{q-1}{q}\right) + \frac{a}{x_0^{q-1}}\left(\frac{1}{q}\right) = \frac{a}{x_0^{q-1}}\left(\frac{q-1}{q} + \frac{1}{q}\right) = \frac{a}{x_0^{q-1}}$.

22. We have that $(x-h)^2 + (y-h)^2 = r^2$ and so $2(x-h) + 2(y-h)\frac{dy}{dx} = 0$ and $2 + 2\frac{dy}{dx} + 2(y-h)\frac{d^2y}{dx^2} = 0$ hold. Thus $2x + 2y\frac{dy}{dx} = 2h + 2h\frac{dy}{dx}$, by the former. Solving for h, we obtain $h = \frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}$. Substituting this into the second equation yields $2 + 2\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} - 2\left(\frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right) = 0$. Dividing by 2 results in $1 + \frac{dy}{dx} + y\frac{d^2y}{dx^2} - \left(\frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right) = 0$.

23. (a) $a(t) = s''(t) = -k \ \ (k > 0) \Rightarrow s'(t) = -kt + C_1$, where $s'(0) = 88 \Rightarrow C_1 = 88 \Rightarrow s'(t) = -kt + 88$. So $s(t) = \frac{-kt^2}{2} + 88t + C_2$ where $s(0) = 0 \Rightarrow C_2 = 0$ so $s(t) = \frac{-kt^2}{2} + 88t$. Now $s(t) = 100$ when $\frac{-kt^2}{2} + 88t = 100$. Solving for t we obtain $t = \frac{88 \pm \sqrt{88^2 - 200k}}{k}$. At such t we want $s'(t) = 0$, thus $-k\left(\frac{88 + \sqrt{88^2 - 200k}}{k}\right) + 88 = 0$ or $-k\left(\frac{88 - \sqrt{88^2 - 200k}}{k}\right) + 88 = 0$. In either case we obtain $88^2 - 200k = 0$ so that $k = \frac{88^2}{200} \approx 38.72 \text{ ft/sec}^2$.
(b) The initial condition that $s'(0) = 44$ ft/sec implies that $s'(t) = -kt + 44$ and $s(t) = \frac{-kt^2}{2} + 44t$ where k is as above. The car is stopped at a time t such that $s'(t) = -kt + 44 = 0 \Rightarrow t = \frac{44}{k}$. At this time the car has traveled a distance $s\left(\frac{44}{k}\right) = \frac{-k}{2}\left(\frac{44}{k}\right)^2 + 44\left(\frac{44}{k}\right) = \frac{44^2}{2k} = \frac{968}{k} = 968\left(\frac{200}{88^2}\right) = 25$ feet. Thus halving the initial velocity quarters stopping distance.

24. $h(x) = f^2(x) + g^2(x) \Rightarrow h'(x) = 2f(x)f'(x) + 2g(x)g'(x) = 2\left[f(x)f'(x) + g(x)g'(x)\right] = 2\left[f(x)g(x) + g(x)(-f(x))\right]$
$= 2 \cdot 0 = 0$. Thus $h(x) = c$, a constant. Since $h(0) = 5$, $h(x) = 5$ for all x in the domain of h. Thus $h(10) = 5$.

25. Yes. The curve $y = x$ satisfies all three conditions since $\frac{dy}{dx} = 1$ everywhere, when $x = 0$, $y = 0$, and $\frac{d^2y}{dx^2} = 0$ everywhere.

26. $y' = 3x^2 + 2$ for all $x \Rightarrow y = x^3 + 2x + C$ where $-1 = 1^3 + 2 \cdot 1 + C \Rightarrow C = -4 \Rightarrow y = x^3 + 2x - 4$.

27. $s''(t) = a = -t^2 \Rightarrow v = s'(t) = \frac{-t^3}{3} + C$. We seek $v_0 = s'(0) = C$. We know that $s(t^*) = b$ for some t^* and s is at a maximum for this t^*. Since $s(t) = \frac{-t^4}{12} + Ct + k$ and $s(0) = 0$ we have that $s(t) = \frac{-t^4}{12} + Ct$ and also $s'(t^*) = 0$ so that $t^* = (3C)^{1/3}$. So $\frac{\left[-(3C)^{1/3}\right]^4}{12} + C(3C)^{1/3} = b \Rightarrow (3C)^{1/3}\left(C - \frac{3C}{12}\right) = b \Rightarrow (3C)^{1/3}\left(\frac{3C}{4}\right) = b \Rightarrow 3^{1/3}C^{4/3} = \frac{4b}{3}$
$\Rightarrow C = \frac{(4b)^{3/4}}{3}$. Thus $v_0 = s'(0) = \frac{(4b)^{3/4}}{3} = \frac{2\sqrt{2}}{3}b^{3/4}$.

28. (a) $s''(t) = t^{1/2} - t^{-1/2} \Rightarrow v(t) = s'(t) = \frac{2}{3}t^{3/2} - 2t^{1/2} + k$ where $v(0) = k = \frac{4}{3} \Rightarrow v(t) = \frac{2}{3}t^{3/2} - 2t^{1/2} + \frac{4}{3}$.
(b) $s(t) = \frac{4}{15}t^{5/2} - \frac{4}{3}t^{3/2} + \frac{4}{3}t + k_2$ where $s(0) = k_2 = -\frac{4}{15}$. Thus $s(t) = \frac{4}{15}t^{5/2} - \frac{4}{3}t^{3/2} + \frac{4}{3}t - \frac{4}{15}$.

29. The graph of $f(x) = ax^2 + bx + c$ with $a > 0$ is a parabola opening upwards. Thus $f(x) \geq 0$ for all x if $f(x) = 0$ for at most one real value of x. The solutions to $f(x) = 0$ are, by the quadratic equation $\frac{-2b \pm \sqrt{(2b)^2 - 4ac}}{2a}$. Thus we require $(2b)^2 - 4ac \leq 0 \Rightarrow b^2 - ac \leq 0$.

30. (a) Clearly $f(x) = (a_1x + b_1)^2 + \ldots + (a_nx + b_n)^2 \geq 0$ for all x. Expanding we see

$f(x) = (a_1^2x^2 + 2a_1b_1x + b_1^2) + \ldots + (a_n^2x^2 + 2a_nb_nx + b_n^2)$

$= (a_1^2 + a_2^2 + \ldots + a_n^2)x^2 + 2(a_1b_1 + a_2b_2 + \ \ldots \ + a_nb_n)x + (b_1^2 + b_2^2 + \ \ldots + b_n^2) \geq 0.$

Thus $(a_1b_1 + a_2b_2 + \ \ldots \ + a_nb_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ \ldots + b_n^2) \leq 0$ by Exercise 29.

Thus $(a_1b_1 + a_2b_2 + \ \ldots \ + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ \ldots + b_n^2)$.

(b) Referring to Exercise 29: It is clear that $f(x) = 0$ for some real $x \Leftrightarrow b^2 - 4ac = 0$, by quadratic formula.

Now notice that this implies that

$f(x) = (a_1x + b_1)^2 + \ldots + (a_nx + b_n)^2$

$= (a_1^2 + a_2^2 + \ldots + a_n^2)x^2 + 2(a_1b_1 + a_2b_2 + \ \ldots \ + a_nb_n)x + (b_1^2 + b_2^2 + \ \ldots + b_n^2) = 0$

$\Leftrightarrow (a_1b_1 + a_2b_2 + \ \ldots \ + a_nb_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ \ldots + b_n^2) = 0$

$\Leftrightarrow (a_1b_1 + a_2b_2 + \ \ldots \ + a_nb_n)^2 = (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ \ldots + b_n^2)$

But now $f(x) = 0 \Leftrightarrow a_ix + b_i = 0$ for all $i = 1, 2, \ldots, n \Leftrightarrow a_ix = -b_i = 0$ for all $i = 1, 2, \ldots, n$.

NOTES