CHAPTER 4 APPLICATIONS OF DERIVATIVES

4.1 EXTREME VALUES OF FUNCTIONS

1. An absolute minimum at $x = c_2$, an absolute maximum at $x = b$. Theorem 1 guarantees the existence of such extreme values because h is continuous on $[a, b]$.

2. An absolute minimum at $x = b$, an absolute maximum at $x = c$. Theorem 1 guarantees the existence of such extreme values because f is continuous on $[a, b]$.

3. No absolute minimum. An absolute maximum at $x = c$. Since the function's domain is an open interval, the function does not satisfy the hypotheses of Theorem 1 and need not have absolute extreme values.

4. No absolute extrema. The function is neither continuous nor defined on a closed interval, so it need not fulfill the conclusions of Theorem 1.

5. An absolute minimum at $x = a$ and an absolute maximum at $x = c$. Note that $y = g(x)$ is not continuous but still has extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

6. Absolute minimum at $x = c$ and an absolute maximum at $x = a$. Note that $y = g(x)$ is not continuous but still has absolute extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

7. Local minimum at $(-1, 0)$, local maximum at $(1, 0)$

8. Minima at $(-2, 0)$ and $(2, 0)$, maximum at $(0, 2)$

9. Maximum at $(0, 5)$. Note that there is no minimum since the endpoint $(2, 0)$ is excluded from the graph.

10. Local maximum at $(-3, 0)$, local minimum at $(2, 0)$, maximum at $(1, 2)$, minimum at $(0, -1)$

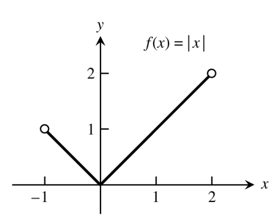
11. Graph (c), since this the only graph that has positive slope at c.

12. Graph (b), since this is the only graph that represents a differentiable function at a and b and has negative slope at c.

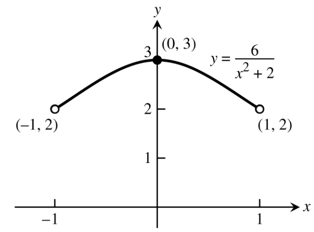
13. Graph (d), since this is the only graph representing a funtion that is differentiable at b but not at a.

14. Graph (a), since this is the only graph that represents a function that is not differentiable at a or b.

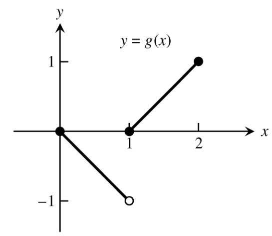
15. f has an absolute min at $x = 0$ but does not have an absolute max. Since the interval on which f is defined, $-1 < x < 2$, is an open interval, we do not meet the conditions of Theorem 1.



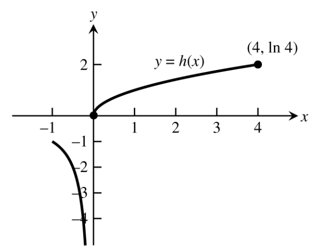
16. f has an absolute max at $x = 0$ but does not have an absolute min. Since the interval on which f is defined, $-1 < x < 1$, is an open interval, we do not meet the conditions of Theorem 1.



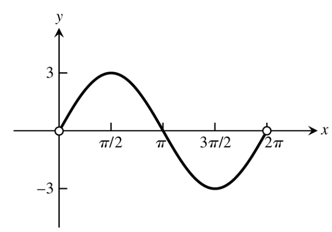
17. f has an absolute max at $x = 2$ but does not have an absolute min. Since the function is not continuous at $x = 1$, we do not meet the conditions of Theorem 1.



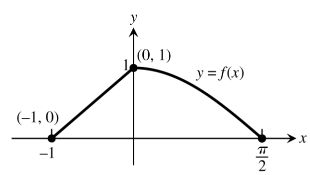
18. f has an absolute max at $x = 4$ but does not have an absolute min. Since the function is not continuous at $x = 0$, we do not meet the conditions of Theorem 1.



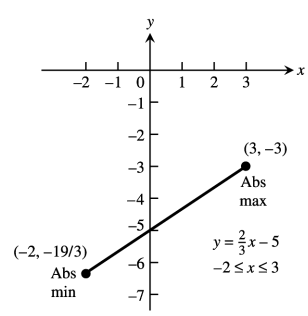
19. f has an absolute max at $x = \frac{\pi}{2}$ and an absolute min at $x = \frac{3\pi}{2}$. Since the interval on which f is defined, $0 < x < 2\pi$, is an open interval, we do not meet the conditions of Theorem 1.



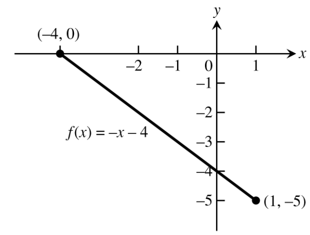
20. f has an absolute max at $x = 0$ and an absolute min at $x = \frac{\pi}{2}$ and $x = -1$. Since f is continuous on the closed interval on which it is defined, $-1 \le x \le 2\pi$, we do meet the conditions of Theorem 1.



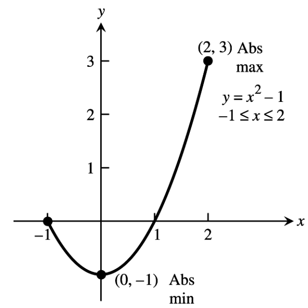
21. $f(x) = \frac{2}{3}x - 5 \Rightarrow f'(x) = \frac{2}{3} \Rightarrow$ no critical points; $f(-2) = -\frac{19}{3}$, $f(3) = -3 \Rightarrow$ the absolute maximum is -3 at $x = 3$ and the absolute minimum is $-\frac{19}{3}$ at $x = -2$



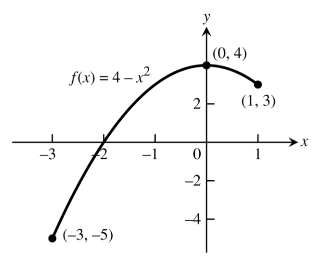
22. $f(x) = -x - 4 \Rightarrow f'(x) = -1 \Rightarrow$ no critical points; $f(-4) = 0$, $f(1) = -5 \Rightarrow$ the absolute maximum is 0 at $x = -4$ and the absolute minimum is -5 at $x = 1$



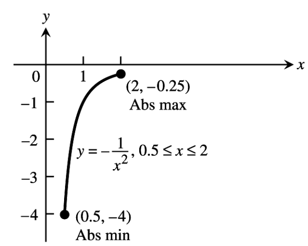
23. $f(x) = x^2 - 1 \Rightarrow f'(x) = 2x \Rightarrow$ a critical point at $x = 0$; $f(-1) = 0$, $f(0) = -1$, $f(2) = 3 \Rightarrow$ the absolute maximum is 3 at $x = 2$ and the absolute minimum is -1 at $x = 0$



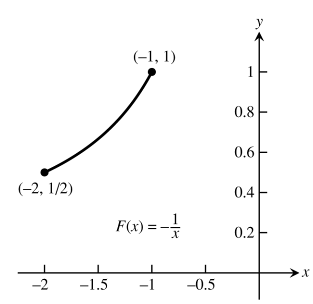
24. $f(x) = 4 - x^2 \Rightarrow f'(x) = -2x \Rightarrow$ a critical point at $x = 0$; $f(-3) = -5$, $f(0) = 4$, $f(1) = 3 \Rightarrow$ the absolute maximum is 4 at $x = 0$ and the absolute minimum is -5 at $x = -3$



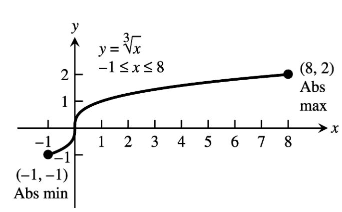
25. $F(x) = -\frac{1}{x^2} = -x^{-2} \Rightarrow F'(x) = 2x^{-3} = \frac{2}{x^3}$, however $x = 0$ is not a critical point since 0 is not in the domain; $F(0.5) = -4$, $F(2) = -0.25 \Rightarrow$ the absolute maximum is -0.25 at $x = 2$ and the absolute minimum is -4 at $x = 0.5$



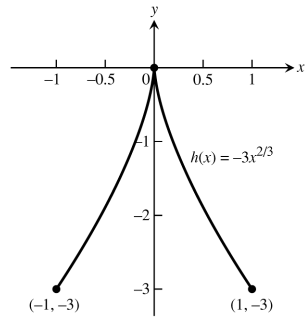
26. $F(x) = -\frac{1}{x} = -x^{-1} \Rightarrow F'(x) = x^{-2} = \frac{1}{x^2}$, however $x = 0$ is not a critical point since 0 is not in the domain; $F(-2) = \frac{1}{2}$, $F(-1) = 1 \Rightarrow$ the absolute maximum is 1 at $x = -1$ and the absolute minimum is $\frac{1}{2}$ at $x = -2$



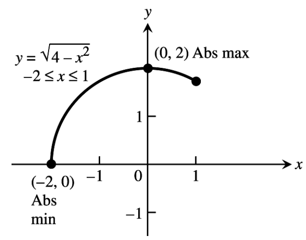
27. $h(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow h'(x) = \frac{1}{3}x^{-2/3} \Rightarrow$ a critical point at $x = 0$; $h(-1) = -1$, $h(0) = 0$, $h(8) = 2 \Rightarrow$ the absolute maximum is 2 at $x = 8$ and the absolute minimum is -1 at $x = -1$



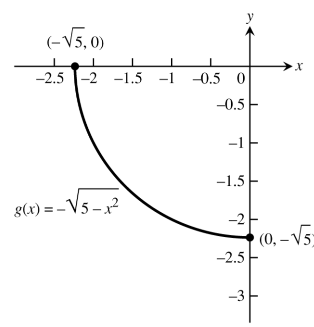
28. $h(x) = -3x^{2/3} \Rightarrow h'(x) = -2x^{-1/3} \Rightarrow$ a critical point at $x = 0$; $h(-1) = -3$, $h(0) = 0$, $h(1) = -3 \Rightarrow$ the absolute maximum is 0 at $x = 0$ and the absolute minimum is -3 at $x = 1$ and at $x = -1$



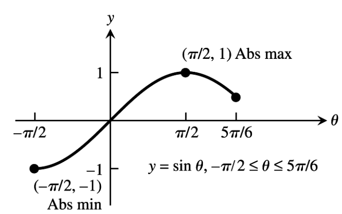
29. $g(x) = \sqrt{4 - x^2} = (4 - x^2)^{1/2}$
$\Rightarrow g'(x) = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$
$\Rightarrow$ critical points at $x = -2$ and $x = 0$, but not at $x = 2$ because 2 is not in the domain; $g(-2) = 0$, $g(0) = 2$, $g(1) = \sqrt{3} \Rightarrow$ the absolute maximum is 2 at $x = 0$ and the absolute minimum is 0 at $x = -2$



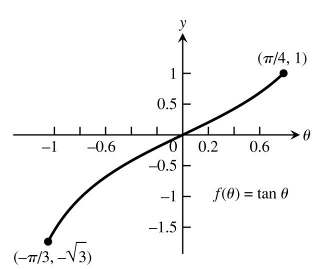
30. $g(x) = -\sqrt{5 - x^2} = -(5 - x^2)^{1/2}\ (5 - x^2)^{-1/2}(-2x)$
$\Rightarrow g'(x) = -\left(\frac{1}{2}\right) = \frac{x}{\sqrt{5 - x^2}} \Rightarrow$ critical points at $x = -\sqrt{5}$ and $x = 0$, but not at $x = \sqrt{5}$ because $\sqrt{5}$ is not in the domain; $f\left(-\sqrt{5}\right) = 0$, $f(0) = -\sqrt{5}$
$\Rightarrow$ the absolute maximum is 0 at $x = -\sqrt{5}$ and the absolute minimum is $-\sqrt{5}$ at $x = 0$



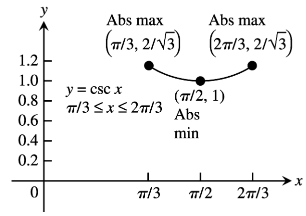
31. $f(\theta) = \sin\theta \Rightarrow f'(\theta) = \cos\theta \Rightarrow \theta = \frac{\pi}{2}$ is a critical point, but $\theta = \frac{-\pi}{2}$ is not a critical point because $\frac{-\pi}{2}$ is not interior to the domain; $f\left(\frac{-\pi}{2}\right) = -1$, $f\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{5\pi}{6}\right) = \frac{1}{2}$
$\Rightarrow$ the absolute maximum is 1 at $\theta = \frac{\pi}{2}$ and the absolute minimum is -1 at $\theta = \frac{-\pi}{2}$



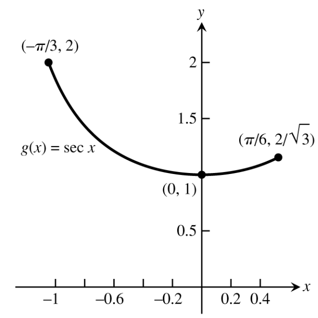
32. $f(\theta) = \tan\theta \Rightarrow f'(\theta) = \sec^2\theta \Rightarrow$ f has no critical points in $\left(\frac{-\pi}{3}, \frac{\pi}{4}\right)$. The extreme values therefore occur at the endpoints: $f\left(\frac{-\pi}{3}\right) = -\sqrt{3}$ and $f\left(\frac{\pi}{4}\right) = 1 \Rightarrow$ the absolute maximum is 1 at $\theta = \frac{\pi}{4}$ and the absolute minimum is $-\sqrt{3}$ at $\theta = \frac{-\pi}{3}$



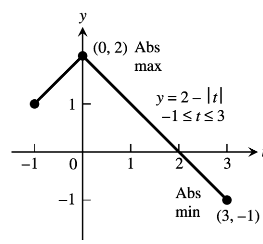
33. $g(x) = \csc x \Rightarrow g'(x) = -(\csc x)(\cot x) \Rightarrow$ a critical point at $x = \frac{\pi}{2}$; $g\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}}$, $g\left(\frac{\pi}{2}\right) = 1$, $g\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}} \Rightarrow$ the absolute maximum is $\frac{2}{\sqrt{3}}$ at $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$, and the absolute minimum is 1 at $x = \frac{\pi}{2}$



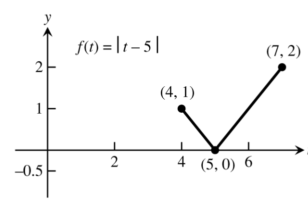
34. $g(x) = \sec x \Rightarrow g'(x) = (\sec x)(\tan x) \Rightarrow$ a critical point at $x = 0$; $g\left(-\frac{\pi}{3}\right) = 2$, $g(0) = 1$, $g\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} \Rightarrow$ the absolute maximum is 2 at $x = -\frac{\pi}{3}$ and the absolute minimum is 1 at $x = 0$



35. $f(t) = 2 - |t| = 2 - \sqrt{t^2} = 2 - \left(t^2\right)^{1/2}$
$\Rightarrow f'(t) = -\frac{1}{2}\left(t^2\right)^{-1/2}(2t) = -\frac{t}{\sqrt{t^2}} = -\frac{t}{|t|}$
$\Rightarrow$ a critical point at $t = 0$; $f(-1) = 1$, $f(0) = 2$, $f(3) = -1 \Rightarrow$ the absolute maximum is 2 at $t = 0$ and the absolute minimum is -1 at $t = 3$



36. $f(t) = |t-5| = \sqrt{(t-5)^2} = \left((t-5)^2\right)^{1/2} \Rightarrow f'(t)$
$= \frac{1}{2}\left((t-5)^2\right)^{-1/2}(2(t-5)) = \frac{t-5}{\sqrt{(t-5)^2}} = \frac{t-5}{|t-5|}$
$\Rightarrow$ a critical point at $t = 5$; $f(4) = 1$, $f(5) = 0$, $f(7) = 2$
$\Rightarrow$ the absolute maximum is 2 at $t = 7$ and the absolute minimum is 0 at $t = 5$



37. $f(x) = x^{4/3} \Rightarrow f'(x) = \frac{4}{3}x^{1/3} \Rightarrow$ a critical point at $x = 0$; $f(-1) = 1$, $f(0) = 0$, $f(8) = 16 \Rightarrow$ the absolute maximum is 16 at $x = 8$ and the absolute minimum is 0 at $x = 0$

38. $f(x) = x^{5/3} \Rightarrow f'(x) = \frac{5}{3}x^{2/3} \Rightarrow$ a critical point at $x = 0$; $f(-1) = -1$, $f(0) = 0$, $f(8) = 32 \Rightarrow$ the absolute maximum is 32 at $x = 8$ and the absolute minimum is -1 at $x = -1$

39. $g(\theta) = \theta^{3/5} \Rightarrow g'(\theta) = \frac{3}{5}\theta^{-2/5} \Rightarrow$ a critical point at $\theta = 0$; $g(-32) = -8$, $g(0) = 0$, $g(1) = 1 \Rightarrow$ the absolute maximum is 1 at $\theta = 1$ and the absolute minimum is -8 at $\theta = -32$

40. $h(\theta) = 3\theta^{2/3} \Rightarrow h'(\theta) = 2\theta^{-1/3} \Rightarrow$ a critical point at $\theta = 0$; $h(-27) = 27$, $h(0) = 0$, $h(8) = 12 \Rightarrow$ the absolute maximum is 27 at $\theta = -27$ and the absolute minimum is 0 at $\theta = 0$

41. $y = x^2 - 6x + 7 \Rightarrow y' = 2x - 6 \Rightarrow 2x - 6 = 0 \Rightarrow x = 3$. The critical point is $x = 3$.

42. $f(x) = 6x^2 - x^3 \Rightarrow f'(x) = 12x - 3x^2 \Rightarrow 12x - 3x^2 = 0 \Rightarrow 3x(4 - x) = 0 \Rightarrow x = 0$ or $x = 4$. The critical pointss are $x = 0$ and $x = 4$.

43. $f(x) = x(4-x)^3 \Rightarrow f'(x) = x\left[3(4-x)^2(-1)\right] + (4-x)^3 = (4-x)^2\left[-3x + (4-x)\right] = (4-x)^2(4-4x)$
$= 4(4-x)^2(1-x) \Rightarrow 4(4-x)^2(1-x) = 0 \Rightarrow x = 1$ or $x = 4$. The critical points are $x = 1$ and $x = 4$.

44. $g(x) = (x-1)^2(x-3)^2 \Rightarrow g'(x) = (x-1)^2 \cdot 2(x-3)(1) + 2(x-1)(1) \cdot (x-3)^2$
$= 2(x-3)(x-1)\left[(x-1) + (x-3)\right] = 4(x-3)(x-1)(x-2) \Rightarrow 4(x-3)(x-1)(x-2) = 0 \Rightarrow x = 3$ or $x = 1$ or $x = 2$. The critical points are $x = 1$, $x = 2$, and $x = 3$.

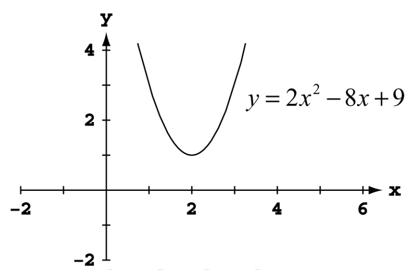
45. $y = x^2 + \frac{2}{x} \Rightarrow y' = 2x - \frac{2}{x^2} = \frac{2x^3-2}{x^2} \Rightarrow \frac{2x^3-2}{x^2} = 0 \Rightarrow 2x^3 - 2 = 0 \Rightarrow x = 1$; $\frac{2x^3-2}{x^2} = \text{undefined} \Rightarrow x^2 = 0 \Rightarrow x = 0$. The domain of the function is $(-\infty, 0) \cup (0, \infty)$, thus $x = 0$ is not in the domain, so the only critical point is $x = 1$.

46. $f(x) = \frac{x^2}{x-2} \Rightarrow f'(x) = \frac{(x-2\cdot)2x - x^2(1)}{(x-2)^2} = \frac{x^2-4x}{(x-2)^2} \Rightarrow \frac{x^2-4x}{(x-2)^2} = 0 \Rightarrow x^2 - 4x = 0 \Rightarrow x = 0$ or $x = 4$; $\frac{x^2-4x}{(x-2)^2} = \text{undefined}$ $\Rightarrow (x-2)^2 = 0 \Rightarrow x = 2$. The domain of the function is $(-\infty, 2) \cup (2, \infty)$, thus $x = 2$ is not in the domain, so the only critical points are $x = 0$ and $x = 4$

47. $y = x^2 - 32\sqrt{x} \Rightarrow y' = 2x - \frac{16}{\sqrt{x}} = \frac{2x^{3/2}-16}{\sqrt{x}} \Rightarrow \frac{2x^{3/2}-16}{\sqrt{x}} = 0 \Rightarrow 2x^{3/2} - 16 = 0 \Rightarrow x = 4$; $\frac{2x^{3/2}-16}{\sqrt{x}} = \text{undefined}$ $\Rightarrow \sqrt{x} = 0 \Rightarrow x = 0$. The critical points are $x = 4$ and $x = 0$.

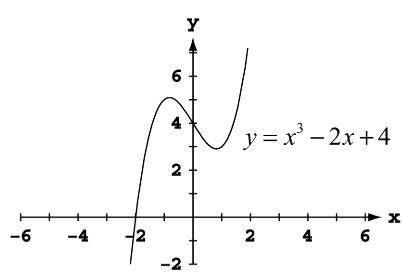
48. $g(x) = \sqrt{2x - x^2} \Rightarrow g'(x) = \frac{1-x}{\sqrt{2x-x^2}} \Rightarrow \frac{1-x}{\sqrt{2x-x^2}} = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$; $\frac{1-x}{\sqrt{2x-x^2}} = \text{undefined} \Rightarrow \sqrt{2x-x^2} = 0$ $2x - x^2 = 0 \Rightarrow x = 0$ or $x = 2$. The critical points are $x = 0$, $x = 1$, and $x = 2$.

49. Minimum value is 1 at $x = 2$.



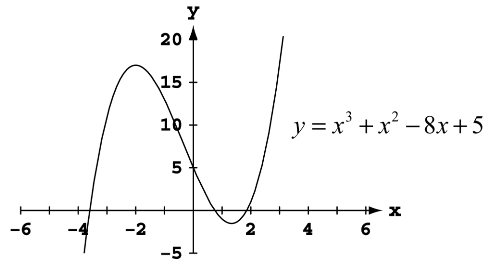
$[-2, 6]$ by $[-2, 4]$

50. To find the exact values, note that $y' = 3x^2 - 2$, which is zero when $x = \pm\sqrt{\frac{2}{3}}$. Local maximum at $\left(-\sqrt{\frac{2}{3}}, 4 + \frac{4\sqrt{6}}{9}\right) \approx (-0.816, 5.089)$; local minimum at $\left(\sqrt{\frac{2}{3}}, 4 - \frac{4\sqrt{6}}{9}\right) \approx (0.816, 2.911)$



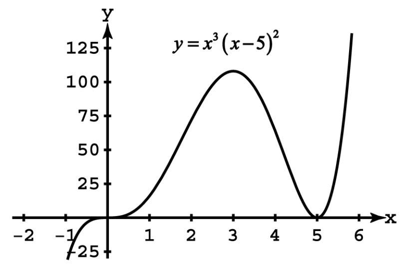
$[-6, 6]$ by $[-2, 7]$

51. To find the exact values, note that that $y' = 3x^2 + 2x - 8$ $= (3x - 4)(x + 2)$, which is zero when $x = -2$ or $x = \frac{4}{3}$. Local maximum at $(-2, 17)$; local minimum at $\left(\frac{4}{3}, -\frac{41}{27}\right)$



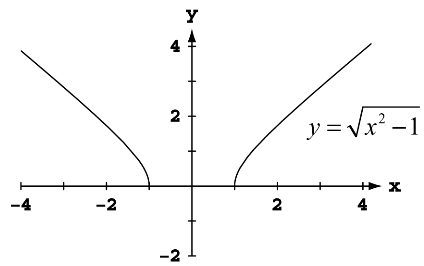
$[-6, 6]$ by $[-5, 20]$

52. Note that $y' = 5x^2(x - 5)(x - 3)$, which is zero at $x = 0$, $x = 3$, and $x = 5$. Local maximum at (3, 108); local minimum at (5, 0); (0, 0) is neither a maximum nor a minimum.



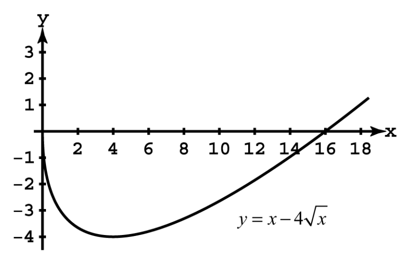
$[-2, 6]$ by $[-25, 125]$

53. Minimum value is 0 when $x = -1$ or $x = 1$.



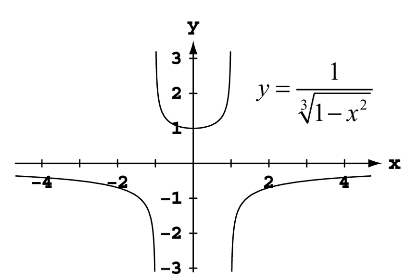
$[-6, 6]$ by $[-2, 4]$

54. Note that $y' = \frac{\sqrt{x} - 2}{\sqrt{x}}$, which is zero at $x = 4$ and is undefined when $x = 0$. Local maximum at (0, 0); absolute minimum at $(4, -4)$



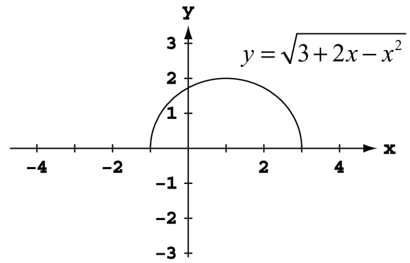
$[-4, 3]$ by $[0, 18]$

55. The actual graph of the function has asymptotes at $x = \pm 1$, so there are no extrema near these values. (This is an example of grapher failure.) There is a local minimum at (0, 1).



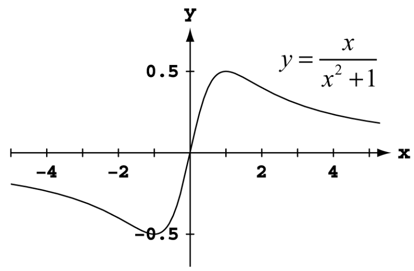
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

56. Maximum value is 2 at $x = 1$;
minimum value is 0 at $x = -1$ and $x = 3$.



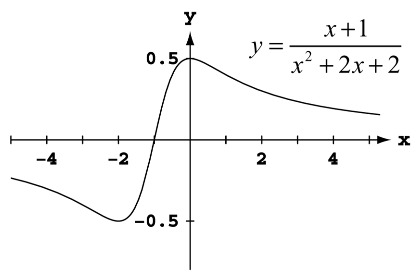
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

57. Maximum value is $\frac{1}{2}$ at $x = 1$;
minimum value is $-\frac{1}{2}$ as $x = -1$.



$[-5, 5]$ by $[-0.7, 0.7]$

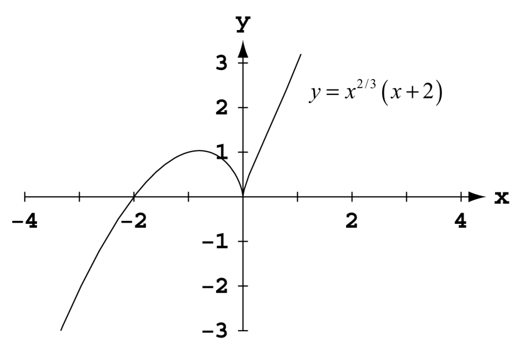
58. Maximum value is $\frac{1}{2}$ at $x = 0$;
minimum value is $-\frac{1}{2}$ as $x = -2$.



$[-5, 5]$ by $[-0.8, 0.6]$

59. $y' = x^{2/3}(1) + \frac{2}{3}x^{-1/3}(x + 2) = \frac{5x + 4}{3\sqrt[3]{x}}$

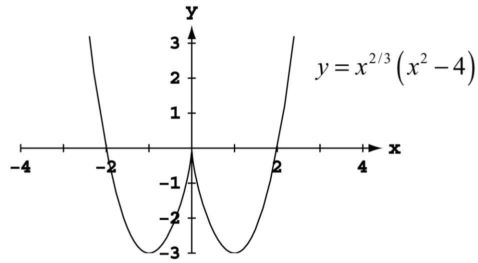
crit. pt.	derivative	extremum	value
$x = -\frac{4}{5}$	0	local max	$\frac{12}{25}10^{1/3} = 1.034$
$x = 0$	undefined	local min	0



$[-4, 4]$ by $[-3, 3]$

60. $y' = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2 - 4) = \frac{8x^2 - 8}{3\sqrt[3]{x}}$

crit. pt.	derivative	extremum	value
$x = -1$	0	minimum	-3
$x = 0$	undefined	local max	0
$x = 1$	0	minimum	3

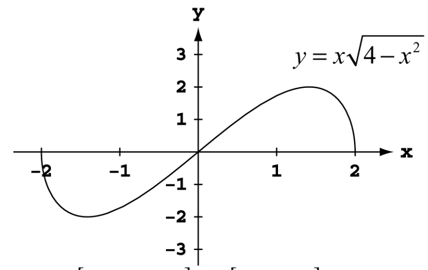


$[-4, 4]$ by $[-3, 3]$

61. $y' = x\frac{1}{2\sqrt{4 - x^2}}(-2x) + (1)\sqrt{4 - x^2}$

$= \frac{-x^2 + (4 - x^2)}{\sqrt{4 - x^2}} = \frac{4 - 2x^2}{\sqrt{4 - x^2}}$

crit. pt.	derivative	extremum	value
$x = -2$	undefined	local max	0
$x = -\sqrt{2}$	0	minimum	-2
$x = \sqrt{2}$	0	maximum	2
$x = 2$	undefined	local min	0

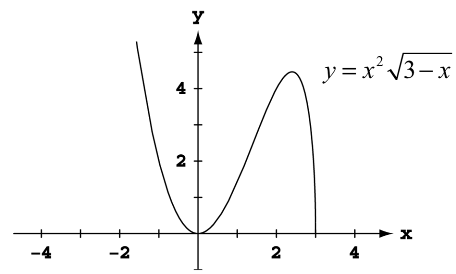


$[-2.35, 2.35]$ by $[-3.5, 3.5]$

62. $y' = x^2\frac{1}{2\sqrt{3 - x}}(-1) + 2x\sqrt{3 - x}$

$= \frac{-x^2 + (4x)(3 - x)}{2\sqrt{3 - x}} = \frac{-5x^2 + 12x}{2\sqrt{3 - x}}$

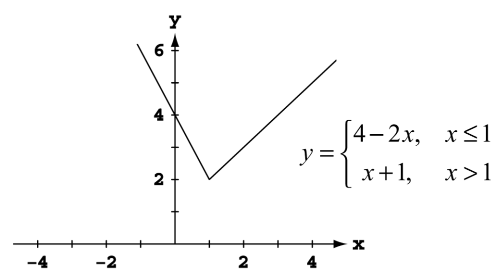
crit. pt.	derivative	extremum	value
$x = 0$	0	minimum	0
$x = \frac{12}{5}$	0	local max	$\frac{144}{125}15^{1/2} \approx 4.462$
$x = 3$	undefined	minimum	0



$[-4.7, 4.7]$ by $[-1, 5]$

63. $y' = \begin{cases} -2, & x < 1 \\ 1, & x > 1 \end{cases}$

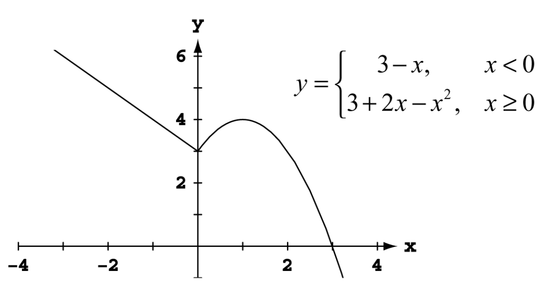
crit. pt.	derivative	extremum	value
$x = 1$	undefined	minimum	2



$[-4.7, 4.7]$ by $[0, 6.2]$

64. $y' = \begin{cases} -1, & x < 0 \\ 2 - 2x, & x > 0 \end{cases}$

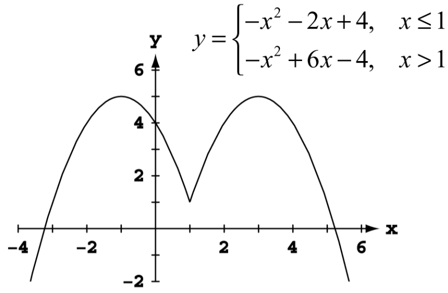
crit. pt.	derivative	extremum	value
$x = 0$	undefined	local min	3
$x = 1$	0	local max	4



$[-4, 4]$ by $[-1, 6]$

65. $y' = \begin{cases} -2x - 2, & x < 1 \\ -2x + 6, & x > 1 \end{cases}$

crit. pt.	derivative	extremum	value
$x = -1$	0	maximum	5
$x = 1$	undefined	local min	1
$x = 3$	0	maximum	5



$[-4, 6]$ by $[-2, 6]$

66. We begin by determining whether $f'(x)$ is defined at $x = 1$, where $f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

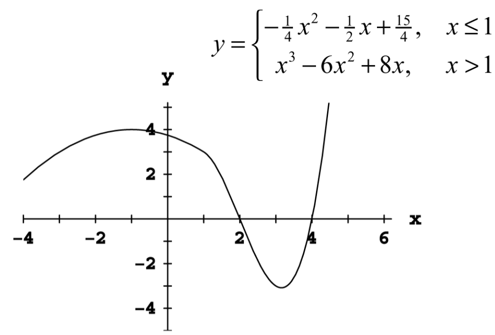
Clearly, $f'(x) = -\frac{1}{2}x - \frac{1}{2}$ if $x < 1$, and $\lim_{h \to 0^-} f'(1 + h) = -1$. Also, $f'(x) = 3x^2 - 12x + 8$ if $x > 1$, and $\lim_{h \to 0^+} f'(1 + h) = -1$. Since f is continuous at $x = 1$, we have that $f'(1) = -1$. Thus,

$f'(x) = \begin{cases} -\frac{1}{2}x - \frac{1}{2}, & x \le 1 \\ 3x^2 - 12x + 8\,, & x > 1 \end{cases}$

Note that $-\frac{1}{2}x - \frac{1}{2} = 0$ when $x = -1$, and $3x^2 - 12x + 8 = 0$ when $x = \frac{12 \pm \sqrt{12^2 - 4(3)(8)}}{2(3)} = \frac{12 \pm \sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$.

But $2 - \frac{2\sqrt{3}}{3} \approx 0.845 < 1$, so the critical points occur at $x = -1$ and $x = 2 + \frac{2\sqrt{3}}{3} \approx 3.155$.

crit. pt.	derivative	extremum	value
$x = -1$	0	local max	4
$x \approx 3.155$	0	local min	≈ -3.079



$[-4, 6]$ by $[-5, 5]$

67. (a) No, since $f'(x) = \frac{2}{3}(x - 2)^{-1/3}$, which is undefined at $x = 2$.
    (b) The derivative is defined and nonzero for all $x \ne 2$. Also, $f(2) = 0$ and $f(x) > 0$ for all $x \ne 2$.
    (c) No, f(x) need not have a global maximum because its domain is all real numbers. Any restriction of f to a closed interval of the form [a, b] would have both a maximum value and minimum value on the interval.
    (d) The answers are the same as (a) and (b) with 2 replaced by a.

68. Note that $f(x) = \begin{cases} -x^3 + 9x, & x \le -3 \text{ or } 0 \le x < 3 \\ x^3 - 9x, & -3 < x < 0 \text{ or } x \ge 3 \end{cases}$. Therefore, $f'(x) = \begin{cases} -3x^3 + 9, & x < -3 \text{ or } 0 < x < 3 \\ 3x^3 - 9, & -3 < x < 0 \text{ or } x > 3 \end{cases}$.
    (a) No, since the left- and right-hand derivatives at $x = 0$, are -9 and 9, respectively.
    (b) No, since the left- and right-hand derivatives at $x = 3$, are -18 and 18, respectively.
    (c) No, since the left- and right-hand derivatives at $x = -3$, are 18 and -18, respectively.
    (d) The critical points occur when $f'(x) = 0$ (at $x = \pm\sqrt{3}$) and when $f'(x)$ is undefined (at $x = 0$ and $x = \pm 3$). The minimum value is 0 at $x = -3$, at $x = 0$, and at $x = 3$; local maxima occur at $\left(-\sqrt{3}, 6\sqrt{3}\right)$ and $\left(\sqrt{3}, 6\sqrt{3}\right)$.

69. Yes, since $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = \frac{x}{(x^2)^{1/2}} = \frac{x}{|x|}$ is not defined at $x = 0$. Thus it is not required that f' be zero at a local extreme point since f' may be undefined there.

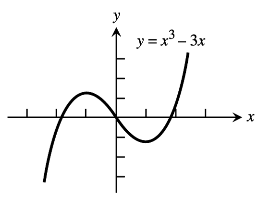
70. If $f(c)$ is a local maximum value of f, then $f(x) \le f(c)$ for all x in some open interval (a, b) containing c. Since f is even, $f(-x) = f(x) \le f(c) = f(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, f assumesa local maximum at the point $-c$. This is also clear from the graph of f because the graph of an even function is symmetric about the y-axis.

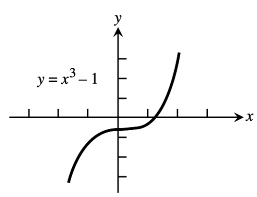
71. If $g(c)$ is a local minimum value of g, then $g(x) \ge g(c)$ for all x in some open interval (a, b) containing c. Since g is odd, $g(-x) = -g(x) \le -g(c) = g(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, g assumes a local maximum at the point $-c$. This is also clear from the graph of g because the graph of an odd function is symmetric about the origin.

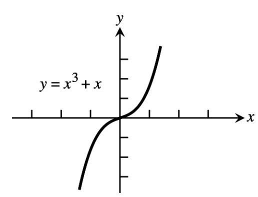
72. If there are no boundary points or critical points the function will have no extreme values in its domain. Such functions do indeed exist, for example $f(x) = x$ for $-\infty < x < \infty$. (Any other linear function $f(x) = mx + b$ with $m \ne 0$ will do as well.)

73. (a) $V(x) = 160x - 52x^2 + 4x^3$
$V'(x) = 160 - 104x + 12x^2 = 4(x - 2)(3x - 20)$
The only critical point in the interval $(0, 5)$ is at $x = 2$. The maximum value of $V(x)$ is 144 at $x = 2$.
(b) The largest possible volume of the box is 144 cubic units, and it occurs when $x = 2$ units.

74. (a) $f'(x) = 3ax^2 + 2bx + c$ is a quadratic, so it can have 0, 1, or 2 zeros, which would be the critical points of f. The function $f(x) = x^3 - 3x$ has two critical points at $x = -1$ and $x = 1$. The function $f(x) = x^3 - 1$ has one critical point at $x = 0$. The function $f(x) = x^3 + x$ has no critical points.





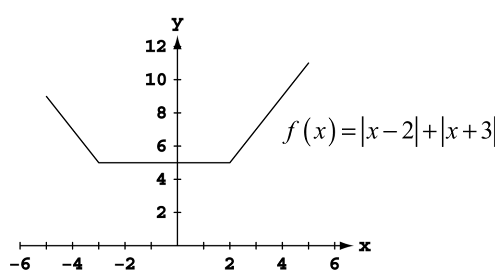


(b) The function can have either two local extreme values or no extreme values. (If there is only one critical point, the cubic function has no extreme values.)

75. $s = -\frac{1}{2}gt^2 + v_0t + s_0 \Rightarrow \frac{ds}{dt} = -gt + v_0 = 0 \Rightarrow t = \frac{v_0}{g}$. Now $s(t) = s_0 \Leftrightarrow t\left(-\frac{gt}{2} + v_0\right) = 0 \Leftrightarrow t = 0$ or $t = \frac{2v_0}{g}$.
Thus $s\left(\frac{v_0}{g}\right) = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) + s_0 = \frac{v_0^2}{2g} + s_0 > s_0$ is the <u>maximum</u> height over the interval $0 \le t \le \frac{2v_0}{g}$.

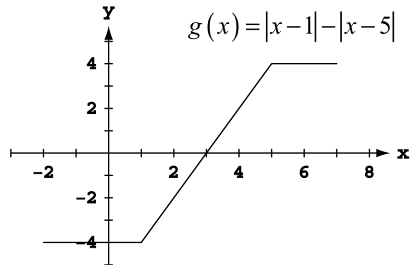
76. $\frac{dI}{dt} = -2\sin t + 2\cos t$, solving $\frac{dI}{dt} = 0 \Rightarrow \tan t = 1 \Rightarrow t = \frac{\pi}{4} + n\pi$ where n is a nonnegative integer (in this exercise t is never negative) $\Rightarrow$ the peak current is $2\sqrt{2}$ amps.

77. Maximum value is 11 at $x = 5$;
minimum value is 5 on the interval $[-3, 2]$;
local maximum at $(-5, 9)$



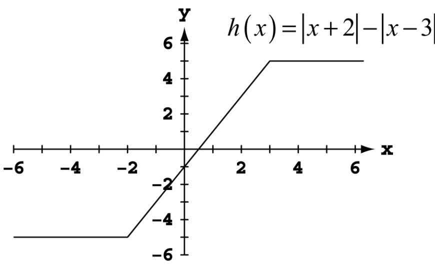
$[-6, 6]$ by $[0, 12]$

78. Maximum value is 4 on the interval $[5,\ 7]$;
minimum value is -4 on the interval $[-2,\ 1]$.



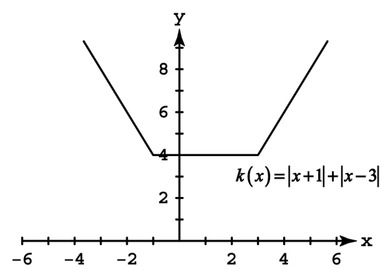
$[-3,8]$ by $[-5,5]$

79. Maximum value is 5 on the interval $[3,\ \infty)$;
minimum value is -5 on the interval $(-\infty,\ -2]$.



$[-6,6]$ by $[-6,6]$

80. Minimum value is 4 on the interval $[-1,\ 3]$



$[-6,6]$ by $[0,9]$

81-86. Example CAS commands:

Maple:

```
with(student):
f := x -> x^4 - 8*x^2 + 4*x + 2;
domain := x=-20/25..64/25;
plot( f(x), domain, color=black, title="Section 4.1 #81(a)" );
Df := D(f);
plot( Df(x), domain, color=black, title="Section 4.1 # 81(b)" )
StatPt := fsolve( Df(x)=0, domain )
SingPt := NULL;
EndPt := op(rhs(domain));
Pts :=evalf([EndPt,StatPt,SingPt]);
Values := [seq( f(x), x=Pts )];
```

Maximum value is 2.7608 and occurs at x=2.56 (right endpoint).

Minimum value [34] is -6.2680 and occurs at x=1.86081 (singular point).

Mathematica: (functions may vary) (see section 2.5 re. RealsOnly):

```
<<Miscellaneous `RealOnly`
Clear[f,x]
a = −1; b = 10/3;
```

```
f[x_] =2 + 2x – 3 x^(2/3)
f'[x]
Plot[{f[x], f'[x]}, {x, a, b}]
NSolve[f'[x]==0, x]
{f[a], f[0], f[x]/.%, f[b]//N
```

In more complicated expressions, NSolve may not yield results. In this case, an approximate solution (say 1.1 here) is observed from the graph and the following command is used:

```
FindRoot[f'[x]==0,{x, 1.1}]
```

4.2 THE MEAN VALUE THEOREM

1. When $f(x) = x^2 + 2x - 1$ for $0 \le x \le 1$, then $\frac{f(1)-f(0)}{1-0} = f'(c) \Rightarrow 3 = 2c + 2 \Rightarrow c = \frac{1}{2}$.

2. When $f(x) = x^{2/3}$ for $0 \le x \le 1$, then $\frac{f(1)-f(0)}{1-0} = f'(c) \Rightarrow 1 = \left(\frac{2}{3}\right) c^{-1/3} \Rightarrow c = \frac{8}{27}$.

3. When $f(x) = x + \frac{1}{x}$ for $\frac{1}{2} \le x \le 2$, then $\frac{f(2)-f(1/2)}{2-1/2} = f'(c) \Rightarrow 0 = 1 - \frac{1}{c^2} \Rightarrow c = 1$.

4. When $f(x) = \sqrt{x-1}$ for $1 \le x \le 3$, then $\frac{f(3)-f(1)}{3-1} = f'(c) \Rightarrow \frac{\sqrt{2}}{2} = \frac{1}{2\sqrt{c-1}} \Rightarrow c = \frac{3}{2}$.

5. When $f(x) = x^3 - x^2$ for $-1 \le x \le 2$, then $\frac{f(2)-f(-1)}{2-(-1)} = f'(c) \Rightarrow 2 = 3c^2 - 2c \Rightarrow c = \frac{1 \pm \sqrt{7}}{3}$.
$\frac{1+\sqrt{7}}{3} \approx 1.22$ and $\frac{1-\sqrt{7}}{3} \approx -0.549$ are both in the interval $-1 \le x \le 2$.

6. When $g(x) = \begin{cases} x^3 & -2 \le x \le 0 \\ x^2 & 0 < x \le 2 \end{cases}$, then $\frac{g(2)-g(-2)}{2-(-2)} = g'(c) \Rightarrow 3 = g'(c)$. If $-2 \le x < 0$, then $g'(x) = 3x^2 \Rightarrow 3 = g'(c)$ $\Rightarrow 3c^2 = 3 \Rightarrow c = \pm 1$. Only $c = -1$ is in the interval. If $0 < x \le 2$, then $g'(x) = 2x \Rightarrow 3 = g'(c) \Rightarrow 2c = 3 \Rightarrow c = \frac{3}{2}$.

7. Does not; $f(x)$ is not differentiable at $x = 0$ in $(-1, 8)$.

8. Does; $f(x)$ is continuous for every point of $[0, 1]$ and differentiable for every point in $(0, 1)$.

9. Does; $f(x)$ is continuous for every point of $[0, 1]$ and differentiable for every point in $(0, 1)$.

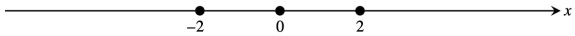
10. Does not; $f(x)$ is not continuous at $x = 0$ because $\lim_{x \to 0^-} f(x) = 1 \ne 0 = f(0)$.

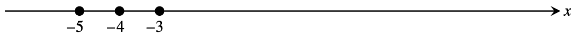
11. Does not; f is not differentiable at $x = -1$ in $(-2, 0)$.

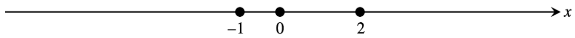
12. Does; $f(x)$ is continuous for every point of $[0, 3]$ and differentiable for every point in $(0, 3)$.

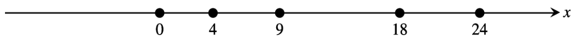
13. Since $f(x)$ is not continuous on $0 \le x \le 1$, Rolle's Theorem does not apply: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1 \ne 0 = f(1)$.

14. Since $f(x)$ must be continuous at $x = 0$ and $x = 1$ we have $\lim_{x \to 0^+} f(x) = a = f(0) \Rightarrow a = 3$ and $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \Rightarrow -1 + 3 + a = m + b \Rightarrow 5 = m + b$. Since $f(x)$ must also be differentiable at $x = 1$ we have $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x) \Rightarrow -2x + 3\big|_{x=1} = m\big|_{x=1} \Rightarrow 1 = m$. Therefore, $a = 3$, $m = 1$ and $b = 4$.

15. (a) i 

ii 

iii 

iv 

(b) Let r_1 and r_2 be zeros of the polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then $P(r_1) = P(r_2) = 0$. Since polynomials are everywhere continuous and differentiable, by Rolle's Theorem $P'(r) = 0$ for some r between r_1 and r_2, where $P'(x) = nx^{n-1} + (n-1)\,a_{n-1}x^{n-2} + \ldots + a_1$.

16. With f both differentiable and continuous on $[a, b]$ and $f(r_1) = f(r_2) = f(r_3) = 0$ where r_1, r_2 and r_3 are in $[a, b]$, then by Rolle's Theorem there exists a c_1 between r_1 and r_2 such that $f'(c_1) = 0$ and a c_2 between r_2 and r_3 such that $f'(c_2) = 0$. Since f' is both differentiable and continuous on $[a, b]$, Rolle's Theorem again applies and we have a c_3 between c_1 and c_2 such that $f''(c_3) = 0$. To generalize, if f has n+1 zeros in $[a, b]$ and $f^{(n)}$ is continuous on $[a, b]$, then $f^{(n)}$ has at least one zero between a and b.

17. Since f'' exists throughout $[a, b]$ the derivative function f' is continuous there. If f' has more than one zero in $[a, b]$, say $f'(r_1) = f'(r_2) = 0$ for $r_1 \neq r_2$, then by Rolle's Theorem there is a c between r_1 and r_2 such that $f''(c) = 0$, contrary to $f'' > 0$ throughout $[a, b]$. Therefore f' has at most one zero in $[a, b]$. The same argument holds if $f'' < 0$ throughout $[a, b]$.

18. If f(x) is a cubic polynomial with four or more zeros, then by Rolle's Theorem $f'(x)$ has three or more zeros, $f''(x)$ has 2 or more zeros and $f'''(x)$ has at least one zero. This is a contradiction since $f'''(x)$ is a non-zero constant when f(x) is a cubic polynomial.

19. With $f(-2) = 11 > 0$ and $f(-1) = -1 < 0$ we conclude from the Intermediate Value Theorem that $f(x) = x^4 + 3x + 1$ has at least one zero between -2 and -1. Then $-2 < x < -1 \Rightarrow -8 < x^3 < -1 \Rightarrow -32 < 4x^3 < -4$ $\Rightarrow -29 < 4x^3 + 3 < -1 \Rightarrow f'(x) < 0$ for $-2 < x < -1 \Rightarrow f(x)$ is decreasing on $[-2, -1] \Rightarrow f(x) = 0$ has exactly one solution in the interval $(-2, -1)$.

20. $f(x) = x^3 + \frac{4}{x^2} + 7 \;\Rightarrow\; f'(x) = 3x^2 - \frac{8}{x^3} > 0$ on $(-\infty, 0) \;\Rightarrow\;$ f(x) is increasing on $(-\infty, 0)$. Also, $f(x) < 0$ if $x < -2$ and $f(x) > 0$ if $-2 < x < 0 \;\Rightarrow\;$ f(x) has exactly one zero in $(-\infty, 0)$.

21. $g(t) = \sqrt{t} + \sqrt{t+1} - 4 \;\Rightarrow\; g'(t) = \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{t+1}} > 0 \;\Rightarrow\;$ g(t) is increasing for t in $(0, \infty)$; $g(3) = \sqrt{3} - 2 < 0$ and $g(15) = \sqrt{15} > 0 \;\Rightarrow\;$ g(t) has exactly one zero in $(0, \infty)$.

22. $g(t) = \frac{1}{1-t} + \sqrt{1+t} - 3.1 \Rightarrow g'(t) = \frac{1}{(1-t)^2} + \frac{1}{2\sqrt{1+t}} > 0 \Rightarrow$ g(t) is increasing for t in $(-1, 1)$; $g(-0.99) = -2.5$ and $g(0.99) = 98.3 \Rightarrow$ g(t) has exactly one zero in $(-1, 1)$.

23. $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8 \;\Rightarrow\; r'(\theta) = 1 + \frac{2}{3}\sin\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right) = 1 + \frac{1}{3}\sin\left(\frac{2\theta}{3}\right) > 0$ on $(-\infty, \infty) \;\Rightarrow\; r(\theta)$ is increasing on $(-\infty, \infty)$; $r(0) = -8$ and $r(8) = \sin^2\left(\frac{8}{3}\right) > 0 \;\Rightarrow\; r(\theta)$ has exactly one zero in $(-\infty, \infty)$.

24. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2} \;\Rightarrow\; r'(\theta) = 2 + 2\sin\theta\cos\theta = 2 + \sin 2\theta > 0$ on $(-\infty, \infty) \;\Rightarrow\; r(\theta)$ is increasing on $(-\infty, \infty)$; $r(-2\pi) = -4\pi - \cos(-2\pi) + \sqrt{2} = -4\pi - 1 + \sqrt{2} < 0$ and $r(2\pi) = 4\pi - 1 + \sqrt{2} > 0 \Rightarrow r(\theta)$ has exactly one zero in $(-\infty, \infty)$.

25. $r(\theta) = \sec\theta - \frac{1}{\theta^3} + 5 \;\Rightarrow\; r'(\theta) = (\sec\theta)(\tan\theta) + \frac{3}{\theta^4} > 0$ on $\left(0, \frac{\pi}{2}\right) \;\Rightarrow\; r(\theta)$ is increasing on $\left(0, \frac{\pi}{2}\right)$; $r(0.1) \approx -994$ and $r(1.57) \approx 1260.5 \;\Rightarrow\; r(\theta)$ has exactly one zero in $\left(0, \frac{\pi}{2}\right)$.

26. $r(\theta) = \tan\theta - \cot\theta - \theta \Rightarrow r'(\theta) = \sec^2\theta + \csc^2\theta - 1 = \sec^2\theta + \cot^2\theta > 0$ on $\left(0, \frac{\pi}{2}\right) \Rightarrow r(\theta)$ is increasing on $\left(0, \frac{\pi}{2}\right)$; $r\left(\frac{\pi}{4}\right) = -\frac{\pi}{4} < 0$ and $r(1.57) \approx 1254.2 \Rightarrow r(\theta)$ has exactly one zero in $\left(0, \frac{\pi}{2}\right)$.

27. By Corollary 1, $f'(x) = 0$ for all $x \Rightarrow f(x) = C$, where C is a constant. Since $f(-1) = 3$ we have $C = 3 \Rightarrow f(x) = 3$ for all x.

28. $g(x) = 2x + 5 \Rightarrow g'(x) = 2 = f'(x)$ for all x. By Corollary 2, $f(x) = g(x) + C$ for some constant C. Then $f(0) = g(0) + C \Rightarrow 5 = 5 + C \Rightarrow C = 0 \Rightarrow f(x) = g(x) = 2x + 5$ for all x.

29. $g(x) = x^2 \Rightarrow g'(x) = 2x = f'(x)$ for all x. By Corollary 2, $f(x) = g(x) + C$.
(a) $f(0) = 0 \Rightarrow 0 = g(0) + C = 0 + C \Rightarrow C = 0 \Rightarrow f(x) = x^2 \Rightarrow f(2) = 4$
(b) $f(1) = 0 \Rightarrow 0 = g(1) + C = 1 + C \Rightarrow C = -1 \Rightarrow f(x) = x^2 - 1 \Rightarrow f(2) = 3$
(c) $f(-2) = 3 \Rightarrow 3 = g(-2) + C \Rightarrow 3 = 4 + C \Rightarrow C = -1 \Rightarrow f(x) = x^2 - 1 \Rightarrow f(2) = 3$

30. $g(x) = mx \Rightarrow g'(x) = m$, a constant. If $f'(x) = m$, then by Corollary 2, $f(x) = g(x) + b = mx + b$ where b is a constant. Therefore all functions whose derivatives are constant can be graphed as straight lines $y = mx + b$.

31. (a) $y = \frac{x^2}{2} + C$ (b) $y = \frac{x^3}{3} + C$ (c) $y = \frac{x^4}{4} + C$

32. (a) $y = x^2 + C$ (b) $y = x^2 - x + C$ (c) $y = x^3 + x^2 - x + C$

33. (a) $y' = -x^{-2} \Rightarrow y = \frac{1}{x} + C$ (b) $y = x + \frac{1}{x} + C$ (c) $y = 5x - \frac{1}{x} + C$

34. (a) $y' = \frac{1}{2}x^{-1/2} \Rightarrow y = x^{1/2} + C \Rightarrow y = \sqrt{x} + C$ (b) $y = 2\sqrt{x} + C$
(c) $y = 2x^2 - 2\sqrt{x} + C$

35. (a) $y = -\frac{1}{2}\cos 2t + C$ (b) $y = 2\sin\frac{t}{2} + C$
(c) $y = -\frac{1}{2}\cos 2t + 2\sin\frac{t}{2} + C$

36. (a) $y = \tan\theta + C$ (b) $y' = \theta^{1/2} \Rightarrow y = \frac{2}{3}\theta^{3/2} + C$ (c) $y = \frac{2}{3}\theta^{3/2} - \tan\theta + C$

37. $f(x) = x^2 - x + C$; $0 = f(0) = 0^2 - 0 + C \Rightarrow C = 0 \Rightarrow f(x) = x^2 - x$

38. $g(x) = -\frac{1}{x} + x^2 + C$; $1 = g(-1) = -\frac{1}{-1} + (-1)^2 + C \Rightarrow C = -1 \Rightarrow g(x) = -\frac{1}{x} + x^2 - 1$

39. $r(\theta) = 8\theta + \cot\theta + C$; $0 = r\left(\frac{\pi}{4}\right) = 8\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right) + C \Rightarrow 0 = 2\pi + 1 + C \Rightarrow C = -2\pi - 1$
$\Rightarrow r(\theta) = 8\theta + \cot\theta - 2\pi - 1$

40. $r(t) = \sec t - t + C$; $0 = r(0) = \sec(0) - 0 + C \Rightarrow C = -1 \Rightarrow r(t) = \sec t - t - 1$

41. $v = \frac{ds}{dt} = 9.8t + 5 \Rightarrow s = 4.9t^2 + 5t + C$; at $s = 10$ and $t = 0$ we have $C = 10 \Rightarrow s = 4.9t^2 + 5t + 10$

42. $v = \frac{ds}{dt} = 32t - 2 \Rightarrow s = 16t^2 - 2t + C$; at $s = 4$ and $t = \frac{1}{2}$ we have $C = 1 \Rightarrow s = 16t^2 - 2t + 1$

43. $v = \frac{ds}{dt} = \sin(\pi t) \Rightarrow s = -\frac{1}{\pi}\cos(\pi t) + C$; at $s = 0$ and $t = 0$ we have $C = \frac{1}{\pi} \Rightarrow s = \frac{1 - \cos(\pi t)}{\pi}$

44. $v = \frac{ds}{dt} = \frac{2}{\pi}\cos\left(\frac{2t}{\pi}\right) \Rightarrow s = \sin\left(\frac{2t}{\pi}\right) + C$; at $s = 1$ and $t = \pi^2$ we have $C = 1 \Rightarrow s = \sin\left(\frac{2t}{\pi}\right) + 1$

45. $a = 32 \Rightarrow v = 32t + C_1$; at $v = 20$ and $t = 0$ we have $C_1 = 20 \Rightarrow v = 32t + 20 \Rightarrow s = 16t^2 + 20t + C_2$; at $s = 5$ and $t = 0$ we have $C_2 = 5 \Rightarrow s = 16t^2 + 20t + 5$

46. $a = 9.8 \Rightarrow v = 9.8t + C_1$; at $v = -3$ and $t = 0$ we have $C_1 = -3 \Rightarrow v = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C_2$; at $s = 0$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = 4.9t^2 - 3t$

47. $a = -4\sin(2t) \Rightarrow v = 2\cos(2t) + C_1$; at $v = 2$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = 2\cos(2t) \Rightarrow s = \sin(2t) + C_2$; at $s = -3$ and $t = 0$ we have $C_2 = -3 \Rightarrow s = \sin(2t) - 3$

48. $a = \frac{9}{\pi^2}\cos\left(\frac{3t}{\pi}\right) \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right) + C_1$; at $v = 0$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right) \Rightarrow s = -\cos\left(\frac{3t}{\pi}\right) + C_2$; at $s = -1$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = -\cos\left(\frac{3t}{\pi}\right)$

49. If $T(t)$ is the temperature of the thermometer at time t, then $T(0) = -19°$ C and $T(14) = 100°$ C. From the Mean Value Theorem there exists a $0 < t_0 < 14$ such that $\frac{T(14) - T(0)}{14 - 0} = 8.5°$ C/sec $= T'(t_0)$, the rate at which the temperature was changing at $t = t_0$ as measured by the rising mercury on the thermometer.

50. Because the trucker's average speed was 79.5 mph, by the Mean Value Theorem, the trucker must have been going that speed at least once during the trip.

51. Because its average speed was approximately 7.667 knots, and by the Mean Value Theorem, it must have been going that speed at least once during the trip.

52. The runner's average speed for the marathon was approximately 11.909 mph. Therefore, by the Mean Value Theorem, the runner must have been going that speed at least once during the marathon. Since the initial speed and final speed are both 0 mph and the runner's speed is continuous, by the Intermediate Value Theorem, the runner's speed must have been 11 mph at least twice.

53. Let $d(t)$ represent the distance the automobile traveled in time t. The average speed over $0 \le t \le 2$ is $\frac{d(2) - d(0)}{2 - 0}$. The Mean Value Theorem says that for some $0 < t_0 < 2$, $d'(t_0) = \frac{d(2) - d(0)}{2 - 0}$. The value $d'(t_0)$ is the speed of the automobile at time t_0 (which is read on the speedometer).

54. $a(t) = v'(t) = 1.6 \Rightarrow v(t) = 1.6t + C$; at $(0, 0)$ we have $C = 0 \Rightarrow v(t) = 1.6t$. When $t = 30$, then $v(30) = 48$ m/sec.

55. The conclusion of the Mean Value Theorem yields $\frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{c^2} \Rightarrow c^2\left(\frac{a-b}{ab}\right) = a - b \Rightarrow c = \sqrt{ab}$.

56. The conclusion of the Mean Value Theorem yields $\frac{b^2 - a^2}{b - a} = 2c \Rightarrow c = \frac{a+b}{2}$.

57. $f'(x) = [\cos x \sin(x+2) + \sin x \cos(x+2)] - 2\sin(x+1)\cos(x+1) = \sin(x + x + 2) - \sin 2(x+1)$
$= \sin(2x+2) - \sin(2x+2) = 0$. Therefore, the function has the constant value $f(0) = -\sin^2 1 \approx -0.7081$ which explains why the graph is a horizontal line.

58. (a) $f(x) = (x+2)(x+1)x(x-1)(x-2) = x^5 - 5x^3 + 4x$ is one possibility.