(c) We say that f(x) approaches minus infinity as x approaches x_0 from the left, and write $\lim_{x \to x_0^-} f(x) = -\infty$, if for every positive number B (or negative number $-B$) there exists a corresponding number $\delta > 0$ such that for all x, $x_0 - \delta < x < x_0 \Rightarrow f(x) < -B$.

94. For $B > 0$, $\frac{1}{x} > B > 0 \Leftrightarrow x < \frac{1}{B}$. Choose $\delta = \frac{1}{B}$. Then $0 < x < \delta \Rightarrow 0 < x < \frac{1}{B} \Rightarrow \frac{1}{x} > B$ so that $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

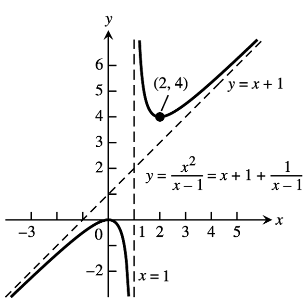
95. For $B > 0$, $\frac{1}{x} < -B < 0 \Leftrightarrow -\frac{1}{x} > B > 0 \Leftrightarrow -x < \frac{1}{B} \Leftrightarrow -\frac{1}{B} < x$. Choose $\delta = \frac{1}{B}$. Then $-\delta < x < 0$ $\Rightarrow -\frac{1}{B} < x \Rightarrow \frac{1}{x} < -B$ so that $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

96. For $B > 0$, $\frac{1}{x-2} < -B \Leftrightarrow -\frac{1}{x-2} > B \Leftrightarrow -(x-2) < \frac{1}{B} \Leftrightarrow x - 2 > -\frac{1}{B} \Leftrightarrow x > 2 - \frac{1}{B}$. Choose $\delta = \frac{1}{B}$. Then $2 - \delta < x < 2 \Rightarrow -\delta < x - 2 < 0 \Rightarrow -\frac{1}{B} < x - 2 < 0 \Rightarrow \frac{1}{x-2} < -B < 0$ so that $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$.

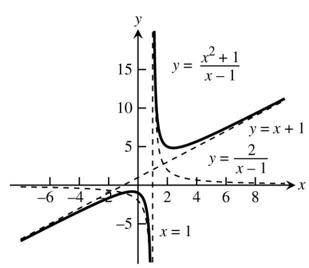
97. For $B > 0$, $\frac{1}{x-2} > B \Leftrightarrow 0 < x - 2 < \frac{1}{B}$. Choose $\delta = \frac{1}{B}$. Then $2 < x < 2 + \delta \Rightarrow 0 < x - 2 < \delta \Rightarrow 0 < x - 2 < \frac{1}{B}$ $\Rightarrow \frac{1}{x-2} > B > 0$ so that $\lim_{x \to 2^+} \frac{1}{x-2} = \infty$.

98. For $B > 0$ and $0 < x < 1$, $\frac{1}{1-x^2} > B \Leftrightarrow 1 - x^2 < \frac{1}{B} \Leftrightarrow (1-x)(1+x) < \frac{1}{B}$. Now $\frac{1+x}{2} < 1$ since $x < 1$. Choose $\delta < \frac{1}{2B}$. Then $1 - \delta < x < 1 \Rightarrow -\delta < x - 1 < 0 \Rightarrow 1 - x < \delta < \frac{1}{2B} \Rightarrow (1-x)(1+x) < \frac{1}{B}\left(\frac{1+x}{2}\right) < \frac{1}{B}$ $\Rightarrow \frac{1}{1-x^2} > B$ for $0 < x < 1$ and x near 1 $\Rightarrow \lim_{x \to 1^-} \frac{1}{1-x^2} = \infty$.

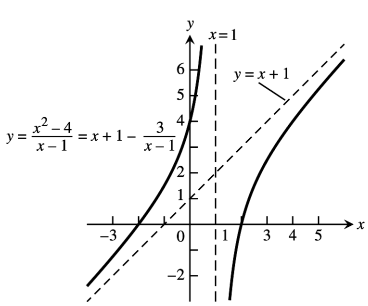
99. $y = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$



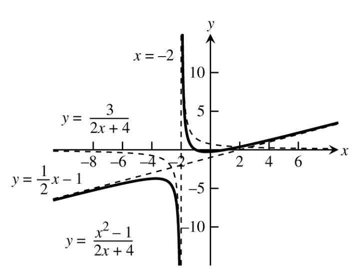
100. $y = \frac{x^2+1}{x-1} = x + 1 + \frac{2}{x-1}$



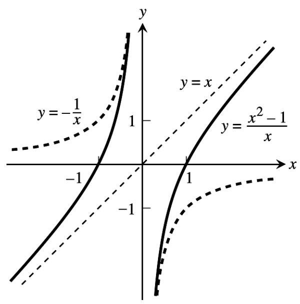
101. $y = \frac{x^2-4}{x-1} = x + 1 - \frac{3}{x-1}$



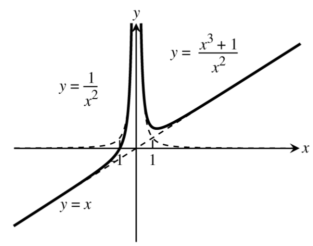
102. $y = \frac{x^2-1}{2x+4} = \frac{1}{2}x - 1 + \frac{3}{2x+4}$



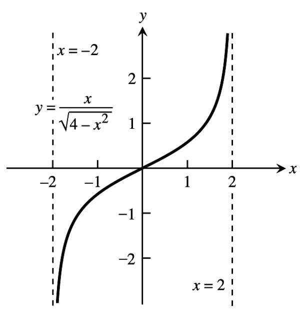
103. $y = \frac{x^2 - 1}{x} = x - \frac{1}{x}$



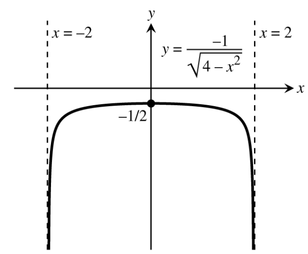
104. $y = \frac{x^3 + 1}{x^2} = x + \frac{1}{x^2}$



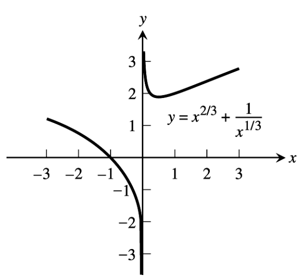
105. $y = \frac{x}{\sqrt{4 - x^2}}$



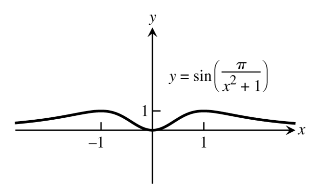
106. $y = \frac{-1}{\sqrt{4 - x^2}}$



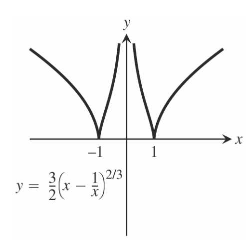
107. $y = x^{2/3} + \frac{1}{x^{1/3}}$



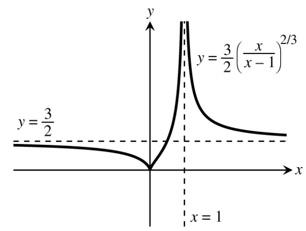
108. $y = \sin\left(\frac{\pi}{x^2 + 1}\right)$



109. (a) $y \to \infty$ (see accompanying graph)
(b) $y \to \infty$ (see accompanying graph)
(c) cusps at $x = \pm 1$ (see accompanying graph)

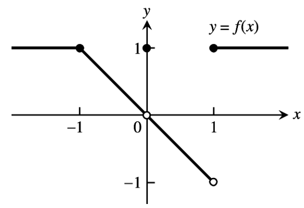


110. (a) $y \to 0$ and a cusp at $x = 0$ (see the accompanying graph)
(b) $y \to \frac{3}{2}$ (see accompanying graph)
(c) a vertical asymptote at $x = 1$ and contains the point $\left(-1, \frac{3}{2\sqrt[3]{4}}\right)$ (see accompanying graph)

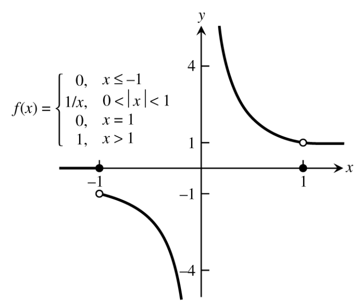


CHAPTER 2 PRACTICE EXERCISES

1. At $x = -1$: $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = 1$
   $\Rightarrow \lim_{x \to -1} f(x) = 1 = f(-1)$
   $\Rightarrow$ f is continuous at $x = -1$.
   At $x = 0$: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0 \Rightarrow \lim_{x \to 0} f(x) = 0$.
   But $f(0) = 1 \neq \lim_{x \to 0} f(x)$
   $\Rightarrow$ f is discontinuous at $x = 0$.
   If we define $f(0) = 0$, then the discontinuity at $x = 0$ is removable.
   At $x = 1$: $\lim_{x \to 1^-} f(x) = -1$ and $\lim_{x \to 1^+} f(x) = 1$
   $\Rightarrow \lim_{x \to 1} f(x)$ does not exist
   $\Rightarrow$ f is discontinuous at $x = 1$.



2. At $x = -1$: $\lim_{x \to -1^-} f(x) = 0$ and $\lim_{x \to -1^+} f(x) = -1$
   $\Rightarrow \lim_{x \to -1} f(x)$ does not exist
   $\Rightarrow$ f is discontinuous at $x = -1$.
   At $x = 0$: $\lim_{x \to 0^-} f(x) = -\infty$ and $\lim_{x \to 0^+} f(x) = \infty$
   $\Rightarrow \lim_{x \to 0} f(x)$ does not exist
   $\Rightarrow$ f is discontinuous at $x = 0$.
   At $x = 1$: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 1 \Rightarrow \lim_{x \to 1} f(x) = 1$.
   But $f(1) = 0 \neq \lim_{x \to 1} f(x)$
   $\Rightarrow$ f is discontinuous at $x = 1$.
   If we define $f(1) = 1$, then the discontinuity at $x = 1$ is removable.



3. (a) $\lim_{t \to t_0} (3f(t)) = 3 \lim_{t \to t_0} f(t) = 3(-7) = -21$
   (b) $\lim_{t \to t_0} (f(t))^2 = \left(\lim_{t \to t_0} f(t)\right)^2 = (-7)^2 = 49$
   (c) $\lim_{t \to t_0} (f(t) \cdot g(t)) = \lim_{t \to t_0} f(t) \cdot \lim_{t \to t_0} g(t) = (-7)(0) = 0$
   (d) $\lim_{t \to t_0} \frac{f(t)}{g(t)-7} = \frac{\lim_{t\to t_0} f(t)}{\lim_{t\to t_0} (g(t) - 7)} = \frac{\lim_{t\to t_0} f(t)}{\lim_{t\to t_0} g(t) - \lim_{t\to t_0} 7} = \frac{-7}{0-7} = 1$
   (e) $\lim_{t \to t_0} \cos(g(t)) = \cos\left(\lim_{t \to t_0} g(t)\right) = \cos 0 = 1$
   (f) $\lim_{t \to t_0} |f(t)| = \left|\lim_{t \to t_0} f(t)\right| = |-7| = 7$
   (g) $\lim_{t \to t_0} (f(t) + g(t)) = \lim_{t \to t_0} f(t) + \lim_{t \to t_0} g(t) = -7 + 0 = -7$
   (h) $\lim_{t \to t_0} \left(\frac{1}{f(t)}\right) = \frac{1}{\lim_{t\to t_0} f(t)} = \frac{1}{-7} = -\frac{1}{7}$

4. (a) $\lim_{x \to 0} -g(x) = -\lim_{x \to 0} g(x) = -\sqrt{2}$
   (b) $\lim_{x \to 0} (g(x) \cdot f(x)) = \lim_{x \to 0} g(x) \cdot \lim_{x \to 0} f(x) = \left(\sqrt{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{2}$
   (c) $\lim_{x \to 0} (f(x) + g(x)) = \lim_{x \to 0} f(x) + \lim_{x \to 0} g(x) = \frac{1}{2} + \sqrt{2}$
   (d) $\lim_{x \to 0} \frac{1}{f(x)} = \frac{1}{\lim_{x\to 0} f(x)} = \frac{1}{\frac{1}{2}} = 2$

(e) $\lim\limits_{x \to 0} (x + f(x)) = \lim\limits_{x \to 0} x + \lim\limits_{x \to 0} f(x) = 0 + \frac{1}{2} = \frac{1}{2}$

(f) $\lim\limits_{x \to 0} \frac{f(x) \cdot \cos x}{x-1} = \frac{\lim_{x\to 0} f(x) \cdot \lim_{x\to 0} \cos x}{\lim_{x\to 0} x - \lim_{x\to 0} 1} = \frac{\left(\frac{1}{2}\right)(1)}{0-1} = -\frac{1}{2}$

5. Since $\lim\limits_{x \to 0} x = 0$ we must have that $\lim\limits_{x \to 0} (4 - g(x)) = 0$. Otherwise, if $\lim\limits_{x \to 0} (4 - g(x))$ is a finite positive number, we would have $\lim\limits_{x \to 0^-} \left[\frac{4-g(x)}{x}\right] = -\infty$ and $\lim\limits_{x \to 0^+} \left[\frac{4-g(x)}{x}\right] = \infty$ so the limit could not equal 1 as $x \to 0$. Similar reasoning holds if $\lim\limits_{x \to 0} (4 - g(x))$ is a finite negative number. We conclude that $\lim\limits_{x \to 0} g(x) = 4$.

6. $2 = \lim\limits_{x \to -4} \left[x \lim\limits_{x \to 0} g(x)\right] = \lim\limits_{x \to -4} x \cdot \lim\limits_{x \to -4} \left[\lim\limits_{x \to 0} g(x)\right] = -4 \lim\limits_{x \to -4} \left[\lim\limits_{x \to 0} g(x)\right] = -4 \lim\limits_{x \to 0} g(x)$
(since $\lim\limits_{x \to 0} g(x)$ is a constant) $\Rightarrow \lim\limits_{x \to 0} g(x) = \frac{2}{-4} = -\frac{1}{2}$.

7. (a) $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} x^{1/3} = c^{1/3} = f(c)$ for every real number c $\Rightarrow$ f is continuous on $(-\infty, \infty)$.
(b) $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} x^{3/4} = c^{3/4} = g(c)$ for every nonnegative real number c $\Rightarrow$ g is continuous on $[0, \infty)$.
(c) $\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c} x^{-2/3} = \frac{1}{c^{2/3}} = h(c)$ for every nonzero real number c $\Rightarrow$ h is continuous on $(-\infty, 0)$ and $(-\infty, \infty)$.
(d) $\lim\limits_{x \to c} k(x) = \lim\limits_{x \to c} x^{-1/6} = \frac{1}{c^{1/6}} = k(c)$ for every positive real number c $\Rightarrow$ k is continuous on $(0, \infty)$

8. (a) $\bigcup\limits_{n \in I} \left(\left(n - \frac{1}{2}\right)\pi, \left(n + \frac{1}{2}\right)\pi\right)$, where I = the set of all integers.
(b) $\bigcup\limits_{n \in I} (n\pi, (n+1)\pi)$, where I = the set of all integers.
(c) $(-\infty, \pi) \cup (\pi, \infty)$
(d) $(-\infty, 0) \cup (0, \infty)$

9. (a) $\lim\limits_{x \to 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x} = \lim\limits_{x \to 0} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim\limits_{x \to 0} \frac{x-2}{x(x+7)}$, $x \neq 2$; the limit does not exist because $\lim\limits_{x \to 0^-} \frac{x-2}{x(x+7)} = \infty$ and $\lim\limits_{x \to 0^+} \frac{x-2}{x(x+7)} = -\infty$
(b) $\lim\limits_{x \to 2} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x} = \lim\limits_{x \to 2} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim\limits_{x \to 2} \frac{x-2}{x(x+7)}$, $x \neq 2$, and $\lim\limits_{x \to 2} \frac{x-2}{x(x+7)} = \frac{0}{2(9)} = 0$

10. (a) $\lim\limits_{x \to 0} \frac{x^2 + x}{x^5 + 2x^4 + x^3} = \lim\limits_{x \to 0} \frac{x(x+1)}{x^3(x^2 + 2x + 1)} = \lim\limits_{x \to 0} \frac{x+1}{x^2(x+1)(x+1)} = \lim\limits_{x \to 0} \frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$.
Now $\lim\limits_{x \to 0^-} \frac{1}{x^2(x+1)} = \infty$ and $\lim\limits_{x \to 0^+} \frac{1}{x^2(x+1)} = \infty \Rightarrow \lim\limits_{x \to 0} \frac{x^2 + x}{x^5 + 2x^4 + x^3} = \infty$.
(b) $\lim\limits_{x \to -1} \frac{x^2 + x}{x^5 + 2x^4 + x^3} = \lim\limits_{x \to -1} \frac{x(x+1)}{x^3(x^2 + 2x + 1)} = \lim\limits_{x \to -1} \frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$. The limit does not exist because $\lim\limits_{x \to -1^-} \frac{1}{x^2(x+1)} = -\infty$ and $\lim\limits_{x \to -1^+} \frac{1}{x^2(x+1)} = \infty$.

11. $\lim\limits_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} = \lim\limits_{x \to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} = \lim\limits_{x \to 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}$

12. $\lim\limits_{x \to a} \frac{x^2 - a^2}{x^4 - a^4} = \lim\limits_{x \to a} \frac{(x^2 - a^2)}{(x^2 + a^2)(x^2 - a^2)} = \lim\limits_{x \to a} \frac{1}{x^2 + a^2} = \frac{1}{2a^2}$

13. $\lim\limits_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim\limits_{h \to 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} = \lim\limits_{h \to 0} (2x + h) = 2x$

14. $\lim\limits_{x \to 0} \frac{(x+h)^2 - x^2}{h} = \lim\limits_{x \to 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} = \lim\limits_{x \to 0} (2x + h) = h$

15. $\lim\limits_{x \to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim\limits_{x \to 0} \frac{2 - (2+x)}{2x(2+x)} = \lim\limits_{x \to 0} \frac{-1}{4 + 2x} = -\frac{1}{4}$

16. $\lim_{x \to 0} \frac{(2+x)^3 - 8}{x} = \lim_{x \to 0} \frac{(x^3 + 6x^2 + 12x + 8) - 8}{x} = \lim_{x \to 0} (x^2 + 6x + 12) = 12$

17. $\lim_{x \to 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1} = \lim_{x \to 1} \frac{(x^{1/3} - 1)}{(\sqrt{x} - 1)} \cdot \frac{(x^{2/3} + x^{1/3} + 1)(\sqrt{x} + 1)}{(\sqrt{x} + 1)(x^{2/3} + x^{1/3} + 1)} = \lim_{x \to 1} \frac{(x-1)(\sqrt{x}+1)}{(x-1)(x^{2/3} + x^{1/3} + 1)} = \lim_{x \to 1} \frac{\sqrt{x}+1}{x^{2/3} + x^{1/3} + 1}$
$= \frac{1+1}{1+1+1} = \frac{2}{3}$

18. $\lim_{x \to 64} \frac{x^{2/3} - 16}{\sqrt{x} - 8} = \lim_{x \to 64} \frac{(x^{1/3} - 4)(x^{1/3} + 4)}{\sqrt{x} - 8} = \lim_{x \to 64} \frac{(x^{1/3} - 4)(x^{1/3} + 4)}{\sqrt{x} - 8} \cdot \frac{(x^{2/3} + 4x^{1/3} + 16)(\sqrt{x} + 8)}{(\sqrt{x} + 8)(x^{2/3} + 4x^{1/3} + 16)}$
$= \lim_{x \to 64} \frac{(x-64)(x^{1/3}+4)(\sqrt{x}+8)}{(x-64)(x^{2/3} + 4x^{1/3} + 16)} = \lim_{x \to 64} \frac{(x^{1/3}+4)(\sqrt{x}+8)}{x^{2/3} + 4x^{1/3} + 16} = \frac{(4+4)(8+8)}{16+16+16} = \frac{8}{3}$

19. $\lim_{x \to 0} \frac{\tan 2x}{\tan \pi x} = \lim_{x \to 0} \frac{\sin 2x}{\cos 2x} \cdot \frac{\cos \pi x}{\sin \pi x} = \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)\left(\frac{\cos \pi x}{\cos 2x}\right)\left(\frac{\pi x}{\sin \pi x}\right)\left(\frac{2x}{\pi x}\right) = 1 \cdot 1 \cdot 1 \cdot \frac{2}{\pi} = \frac{2}{\pi}$

20. $\lim_{x \to \pi^-} \csc x = \lim_{x-\pi^-} \frac{1}{\sin x} = \infty$

21. $\lim_{x \to \pi} \sin\left(\frac{x}{2} + \sin x\right) = \sin\left(\frac{\pi}{2} + \sin \pi\right) = \sin\left(\frac{\pi}{2}\right) = 1$

22. $\lim_{x \to \pi} \cos^2(x - \tan x) = \cos^2(\pi - \tan \pi) = \cos^2(\pi) = (-1)^2 = 1$

23. $\lim_{x \to 0} \frac{8x}{3\sin x - x} = \lim_{x \to 0} \frac{8}{3\frac{\sin x}{x} - 1} = \frac{8}{3(1) - 1} = 4$

24. $\lim_{x \to 0} \frac{\cos 2x - 1}{\sin x} = \lim_{x \to 0} \left(\frac{\cos 2x - 1}{\sin x} \cdot \frac{\cos 2x + 1}{\cos 2x + 1}\right) = \lim_{x \to 0} \frac{\cos^2 2x - 1}{\sin x(\cos 2x + 1)} = \lim_{x \to 0} \frac{-\sin^2 2x}{\sin x(\cos 2x + 1)} = \lim_{x \to 0} \frac{-4\sin x \cos^2 x}{\cos 2x + 1} = \frac{-4(0)(1)^2}{1+1} = 0$

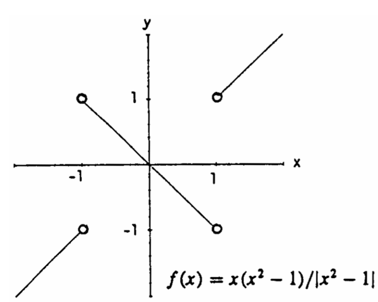
25. $\lim_{x \to 0^+} [4\, g(x)]^{1/3} = 2 \Rightarrow \left[\lim_{x \to 0^+} 4\, g(x)\right]^{1/3} = 2 \Rightarrow \lim_{x \to 0^+} 4\, g(x) = 8$, since $2^3 = 8$. Then $\lim_{x \to 0^+} g(x) = 2$.

26. $\lim_{x \to \sqrt{5}} \frac{1}{x + g(x)} = 2 \Rightarrow \lim_{x \to \sqrt{5}} (x + g(x)) = \frac{1}{2} \Rightarrow \sqrt{5} + \lim_{x \to \sqrt{5}} g(x) = \frac{1}{2} \Rightarrow \lim_{x \to \sqrt{5}} g(x) = \frac{1}{2} - \sqrt{5}$

27. $\lim_{x \to 1} \frac{3x^2 + 1}{g(x)} = \infty \Rightarrow \lim_{x \to 1} g(x) = 0$ since $\lim_{x \to 1} (3x^2 + 1) = 4$

28. $\lim_{x \to -2} \frac{5 - x^2}{\sqrt{g(x)}} = 0 \Rightarrow \lim_{x \to -2} g(x) = \infty$ since $\lim_{x \to -2} (5 - x^2) = 1$

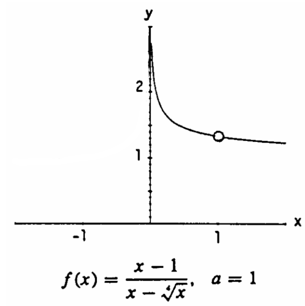
29. At $x = -1$: $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x(x^2-1)}{|x^2-1|}$
$= \lim_{x \to -1^-} \frac{x(x^2-1)}{x^2-1} = \lim_{x \to -1^-} x = -1$, and
$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x \to -1^+} \frac{x(x^2-1)}{-(x^2-1)}$
$= \lim_{x \to -1} (-x) = -(-1) = 1$. Since
$\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$
$\Rightarrow \lim_{x \to -1} f(x)$ does not exist, the function f cannot be extended to a continuous function at $x = -1$.



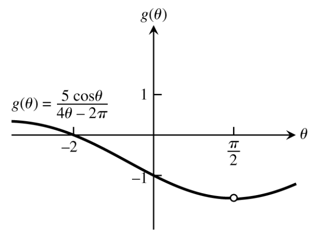
At $x = 1$: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x \to 1^-} \frac{x(x^2-1)}{-(x^2-1)} = \lim_{x \to 1^-} (-x) = -1$, and
$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x \to 1^+} \frac{x(x^2-1)}{x^2-1} = \lim_{x \to 1^+} x = 1$. Again $\lim_{x \to 1} f(x)$ does not exist so f cannot be extended to a continuous function at $x = 1$ either.

30. The discontinuity at $x = 0$ of $f(x) = \sin\left(\frac{1}{x}\right)$ is nonremovable because $\lim\limits_{x \to 0} \sin \frac{1}{x}$ does not exist.

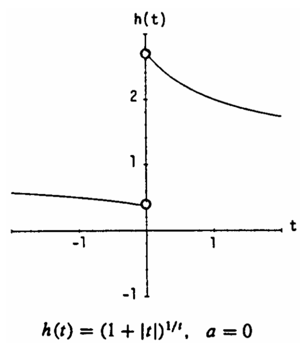
31. Yes, f does have a continuous extension to $a = 1$:
define $f(1) = \lim\limits_{x \to 1} \frac{x-1}{x - \sqrt[4]{x}} = \frac{4}{3}$.



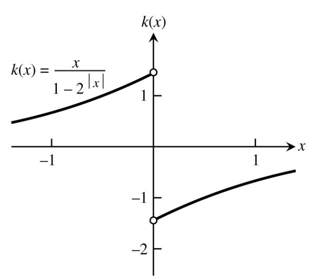
32. Yes, g does have a continuous extension to $a = \frac{\pi}{2}$:
$g\left(\frac{\pi}{2}\right) = \lim\limits_{\theta \to \frac{\pi}{2}} \frac{5\cos\theta}{4\theta - 2\pi} = -\frac{5}{4}$.



33. From the graph we see that $\lim\limits_{t \to 0^-} h(t) \neq \lim\limits_{t \to 0^+} h(t)$
so h <u>cannot</u> be extended to a continuous function
at $a = 0$.



34. From the graph we see that $\lim\limits_{x \to 0^-} k(x) \neq \lim\limits_{x \to 0^+} k(x)$
so k <u>cannot</u> be extended to a continuous function
at $a = 0$.



35. (a) $f(-1) = -1$ and $f(2) = 5 \Rightarrow$ f has a root between -1 and 2 by the Intermediate Value Theorem.
(b), (c) root is 1.32471795724

36. (a) $f(-2) = -2$ and $f(0) = 2 \Rightarrow$ f has a root between -2 and 0 by the Intermediate Value Theorem.
(b), (c) root is -1.76929235424

37. $\lim\limits_{x \to \infty} \frac{2x+3}{5x+7} = \lim\limits_{x \to \infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2+0}{5+0} = \frac{2}{5}$

38. $\lim\limits_{x \to -\infty} \frac{2x^2+3}{5x^2+7} = \lim\limits_{x \to -\infty} \frac{2+\frac{3}{x^2}}{5+\frac{7}{x^2}} = \frac{2+0}{5+0} = \frac{2}{5}$

39. $\lim\limits_{x \to -\infty} \frac{x^2-4x+8}{3x^3} = \lim\limits_{x \to -\infty} \left(\frac{1}{3x} - \frac{4}{3x^2} + \frac{8}{3x^3}\right) = 0 - 0 + 0 = 0$

40. $\lim_{x \to \infty} \frac{1}{x^2 - 7x + 1} = \lim_{x \to \infty} \frac{\frac{1}{x^2}}{1 - \frac{7}{x} + \frac{1}{x^2}} = \frac{0}{1 - 0 + 0} = 0$

41. $\lim_{x \to -\infty} \frac{x^2 - 7x}{x + 1} = \lim_{x \to -\infty} \frac{x - 7}{1 + \frac{1}{x}} = -\infty$

42. $\lim_{x \to \infty} \frac{x^4 + x^3}{12x^3 + 128} = \lim_{x \to -\infty} \frac{x + 1}{12 + \frac{128}{x^3}} = \infty$

43. $\lim_{x \to \infty} \frac{\sin x}{\lfloor x \rfloor} \le \lim_{x \to \infty} \frac{1}{\lfloor x \rfloor} = 0$ since int $x \to \infty$ as $x \to \infty \Rightarrow \lim_{x \to \infty} \frac{\sin x}{\lfloor x \rfloor} = 0$.

44. $\lim_{\theta \to \infty} \frac{\cos\theta - 1}{\theta} \le \lim_{\theta \to \infty} \frac{2}{\theta} = 0 \Rightarrow \lim_{\theta \to \infty} \frac{\cos\theta - 1}{\theta} = 0$.

45. $\lim_{x \to \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x} = \lim_{x \to \infty} \frac{1 + \frac{\sin x}{x} + \frac{2}{\sqrt{x}}}{1 + \frac{\sin x}{x}} = \frac{1 + 0 + 0}{1 + 0} = 1$

46. $\lim_{x \to \infty} \frac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x} = \lim_{x \to \infty} \left(\frac{1 + x^{-5/3}}{1 + \frac{\cos^2 x}{x^{2/3}}} \right) = \frac{1 + 0}{1 + 0} = 1$

47. (a) $y = \frac{x^2 + 4}{x - 3}$ is undefined at $x = 3$: $\lim_{x \to 3^-} \frac{x^2 + 4}{x - 3} = -\infty$ and $\lim_{x \to 3^+} \frac{x^2 + 4}{x - 3} = +\infty$, thus $x = 3$ is a vertical asymptote.

(b) $y = \frac{x^2 - x - 2}{x^2 - 2x + 1}$ is undefined at $x = 1$: $\lim_{x \to 1^-} \frac{x^2 - x - 2}{x^2 - 2x + 1} = -\infty$ and $\lim_{x \to 1^+} \frac{x^2 - x - 2}{x^2 - 2x + 1} = -\infty$, thus $x = 1$ is a vertical asymptote.

(c) $y = \frac{x^2 + x - 6}{x^2 + 2x - 8}$ is undefined at $x = 2$ and -4: $\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 + 2x - 8} = \lim_{x \to 2} \frac{x + 3}{x + 4} = \frac{5}{6}$; $\lim_{x \to -4^-} \frac{x^2 + x - 6}{x^2 + 2x - 8} = \lim_{x \to -4^-} \frac{x + 3}{x + 4} = \infty$ $\lim_{x \to -4^+} \frac{x^2 + x - 6}{x^2 + 2x - 8} = \lim_{x \to -4^+} \frac{x + 3}{x + 4} = -\infty$. Thus $x = -4$ is a vertical asymptote.

48. (a) $y = \frac{1 - x^2}{x^2 + 1}$: $\lim_{x \to \infty} \frac{1 - x^2}{x^2 + 1} = \lim_{x \to \infty} \frac{\frac{1}{x^2} - 1}{1 + \frac{1}{x^2}} = \frac{-1}{1} = -1$ and $\lim_{x \to -\infty} \frac{1 - x^2}{x^2 + 1} = \lim_{x \to -\infty} \frac{\frac{1}{x^2} - 1}{1 + \frac{1}{x^2}} = \frac{-1}{1} = -1$, thus $y = -1$ is a horizontal asymptote.

(b) $y = \frac{\sqrt{x} + 4}{\sqrt{x + 4}}$: $\lim_{x \to \infty} \frac{\sqrt{x} + 4}{\sqrt{x + 4}} = \lim_{x \to \infty} \frac{1 + \frac{4}{\sqrt{x}}}{\sqrt{1 + \frac{4}{x}}} = \frac{1 + 0}{\sqrt{1 + 0}} = 1$, thus $y = 1$ is a horizontal asymptote.

(c) $y = \frac{\sqrt{x^2 + 4}}{x}$: $\lim_{x \to \infty} \frac{\sqrt{x^2 + 4}}{x} = \lim_{x \to \infty} \frac{\sqrt{1 + \frac{4}{x^2}}}{1} = \frac{\sqrt{1 + 0}}{1} = 1$ and $\lim_{x \to -\infty} \frac{\sqrt{x^2 + 4}}{x} = \lim_{x \to -\infty} \frac{\sqrt{1 + \frac{4}{x^2}}}{\frac{x}{\sqrt{x^2}}} = \lim_{x \to -\infty} \frac{\sqrt{1 + \frac{4}{x^2}}}{\frac{x}{-x}}$ $= \lim_{x \to -\infty} \frac{\sqrt{1 + \frac{4}{x^2}}}{-1} = \frac{\sqrt{1 + 0}}{-1} = \frac{1}{-1} = -1$, thus $y = 1$ and $y = -1$ are horizontal asymptotes.

(d) $y = \sqrt{\frac{x^2 + 9}{9x^2 + 1}}$: $\lim_{x \to \infty} \sqrt{\frac{x^2 + 9}{9x^2 + 1}} = \lim_{x \to \infty} \sqrt{\frac{1 + \frac{9}{x^2}}{9 + \frac{1}{x^2}}} = \sqrt{\frac{1 + 0}{9 + 0}} = \frac{1}{3}$ and $\lim_{x \to -\infty} \sqrt{\frac{x^2 + 9}{9x^2 + 1}} = \lim_{x \to -\infty} \sqrt{\frac{1 + \frac{9}{x^2}}{9 + \frac{1}{x^2}}} = \sqrt{\frac{1 + 0}{9 + 0}} = \frac{1}{3}$, thus $y = \frac{1}{3}$ is a horizontal asymptote.

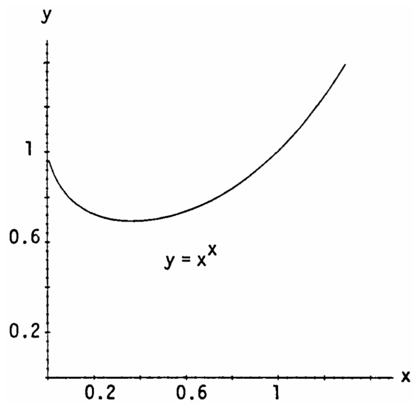
CHAPTER 2 ADDITIONAL AND ADVANCED EXERCISES

1. (a)

x	0.1	0.01	0.001	0.0001	0.00001
x^x	0.7943	0.9550	0.9931	0.9991	0.9999

Apparently, $\lim_{x \to 0^+} x^x = 1$

(b)

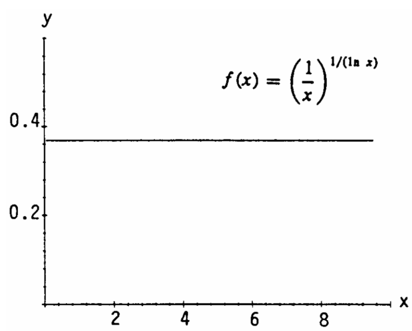


2. (a)

x	10	100	1000
$\left(\frac{1}{x}\right)^{1/(\ln x)}$	0.3679	0.3679	0.3679

Apparently, $\lim_{x \to \infty} \left(\frac{1}{x}\right)^{1/(\ln x)} = 0.3678 = \frac{1}{e}$

(b)



3. $\lim_{v \to c^-} L = \lim_{v \to c^-} L_0 \sqrt{1 - \frac{v^2}{c^2}} = L_0 \sqrt{1 - \frac{\lim_{v \to c^-} v^2}{c^2}} = L_0 \sqrt{1 - \frac{c^2}{c^2}} = 0$
The left-hand limit was needed because the function L is undefined if $v > c$ (the rocket cannot move faster than the speed of light).

4. (a) $\left|\frac{\sqrt{x}}{2} - 1\right| < 0.2 \Rightarrow -0.2 < \frac{\sqrt{x}}{2} - 1 < 0.2 \Rightarrow 0.8 < \frac{\sqrt{x}}{2} < 1.2 \Rightarrow 1.6 < \sqrt{x} < 2.4 \Rightarrow 2.56 < x < 5.76.$
(b) $\left|\frac{\sqrt{x}}{2} - 1\right| < 0.1 \Rightarrow -0.1 < \frac{\sqrt{x}}{2} - 1 < 0.1 \Rightarrow 0.9 < \frac{\sqrt{x}}{2} < 1.1 \Rightarrow 1.8 < \sqrt{x} < 2.2 \Rightarrow 3.24 < x < 4.84.$

5. $|10 + (t - 70) \times 10^{-4} - 10| < 0.0005 \Rightarrow |(t - 70) \times 10^{-4}| < 0.0005 \Rightarrow -0.0005 < (t - 70) \times 10^{-4} < 0.0005$
$\Rightarrow -5 < t - 70 < 5 \Rightarrow 65° < t < 75° \Rightarrow$ Within 5° F.

6. We want to know in what interval to hold values of h to make V satisfy the inequality
$|V - 1000| = |36\pi h - 1000| \le 10$. To find out, we solve the inequality:
$|36\pi h - 1000| \le 10 \Rightarrow -10 \le 36\pi h - 1000 \le 10 \Rightarrow 990 \le 36\pi h \le 1010 \Rightarrow \frac{990}{36\pi} \le h \le \frac{1010}{36\pi}$
$\Rightarrow 8.8 \le h \le 8.9$. where 8.8 was rounded up, to be safe, and 8.9 was rounded down, to be safe.
The interval in which we should hold h is about $8.9 - 8.8 = 0.1$ cm wide (1 mm). With stripes 1 mm wide, we can expect to measure a liter of water with an accuracy of 1%, which is more than enough accuracy for cooking.

7. Show $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 - 7) = -6 = f(1)$.
Step 1: $|(x^2 - 7) + 6| < \epsilon \Rightarrow -\epsilon < x^2 - 1 < \epsilon \Rightarrow 1 - \epsilon < x^2 < 1 + \epsilon \Rightarrow \sqrt{1 - \epsilon} < x < \sqrt{1 + \epsilon}$.

Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow -\delta+1<x<\delta+1$.

Then $-\delta+1=\sqrt{1-\epsilon}$ or $\delta+1=\sqrt{1+\epsilon}$. Choose $\delta=\min\left\{1-\sqrt{1-\epsilon}, \sqrt{1+\epsilon}-1\right\}$, then

$0<|x-1|<\delta \Rightarrow |(x^2-7)-6|<\epsilon$ and $\lim\limits_{x\to 1} f(x)=-6$. By the continuity test, f(x) is continuous at x = 1.

8. Show $\lim\limits_{x\to\frac{1}{4}} g(x)=\lim\limits_{x\to\frac{1}{4}} \frac{1}{2x}=2=g\left(\frac{1}{4}\right)$.

Step 1: $\left|\frac{1}{2x}-2\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{2x}-2<\epsilon \Rightarrow 2-\epsilon<\frac{1}{2x}<2+\epsilon \Rightarrow \frac{1}{4-2\epsilon}>x>\frac{1}{4+2\epsilon}$.

Step 2: $\left|x-\frac{1}{4}\right|<\delta \Rightarrow -\delta<x-\frac{1}{4}<\delta \Rightarrow -\delta+\frac{1}{4}<x<\delta+\frac{1}{4}$.

Then $-\delta+\frac{1}{4}=\frac{1}{4+2\epsilon} \Rightarrow \delta=\frac{1}{4}-\frac{1}{4+2\epsilon}=\frac{\epsilon}{4(2+\epsilon)}$, or $\delta+\frac{1}{4}=\frac{1}{4-2\epsilon} \Rightarrow \delta=\frac{1}{4-2\epsilon}-\frac{1}{4}=\frac{\epsilon}{4(2-\epsilon)}$.

Choose $\delta=\frac{\epsilon}{4(2+\epsilon)}$, the smaller of the two values. Then $0<\left|x-\frac{1}{4}\right|<\delta \Rightarrow \left|\frac{1}{2x}-2\right|<\epsilon$ and $\lim\limits_{x\to\frac{1}{4}} \frac{1}{2x}=2$.

By the continuity test, g(x) is continuous at $x=\frac{1}{4}$.

9. Show $\lim\limits_{x\to 2} h(x)=\lim\limits_{x\to 2} \sqrt{2x-3}=1=h(2)$.

Step 1: $\left|\sqrt{2x-3}-1\right|<\epsilon \Rightarrow -\epsilon<\sqrt{2x-3}-1<\epsilon \Rightarrow 1-\epsilon<\sqrt{2x-3}<1+\epsilon \Rightarrow \frac{(1-\epsilon)^2+3}{2}<x<\frac{(1+\epsilon)^2+3}{2}$.

Step 2: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta$ or $-\delta+2<x<\delta+2$.

Then $-\delta+2=\frac{(1-\epsilon)^2+3}{2} \Rightarrow \delta=2-\frac{(1-\epsilon)^2+3}{2}=\frac{1-(1-\epsilon)^2}{2}=\epsilon-\frac{\epsilon^2}{2}$, or $\delta+2=\frac{(1+\epsilon)^2+3}{2}$

$\Rightarrow \delta=\frac{(1+\epsilon)^2+3}{2}-2=\frac{(1+\epsilon)^2-1}{2}=\epsilon+\frac{\epsilon^2}{2}$. Choose $\delta=\epsilon-\frac{\epsilon^2}{2}$, the smaller of the two values. Then,

$0<|x-2|<\delta \Rightarrow \left|\sqrt{2x-3}-1\right|<\epsilon$, so $\lim\limits_{x\to 2} \sqrt{2x-3}=1$. By the continuity test, h(x) is continuous at x = 2.

10. Show $\lim\limits_{x\to 5} F(x)=\lim\limits_{x\to 5} \sqrt{9-x}=2=F(5)$.

Step 1: $\left|\sqrt{9-x}-2\right|<\epsilon \Rightarrow -\epsilon<\sqrt{9-x}-2<\epsilon \Rightarrow 9-(2-\epsilon)^2>x>9-(2+\epsilon)^2$.

Step 2: $0<|x-5|<\delta \Rightarrow -\delta<x-5<\delta \Rightarrow -\delta+5<x<\delta+5$.

Then $-\delta+5=9-(2+\epsilon)^2 \Rightarrow \delta=(2+\epsilon)^2-4=\epsilon^2+2\epsilon$, or $\delta+5=9-(2-\epsilon)^2 \Rightarrow \delta=4-(2-\epsilon)^2=\epsilon^2-2\epsilon$.

Choose $\delta=\epsilon^2-2\epsilon$, the smaller of the two values. Then, $0<|x-5|<\delta \Rightarrow \left|\sqrt{9-x}-2\right|<\epsilon$, so

$\lim\limits_{x\to 5} \sqrt{9-x}=2$. By the continuity test, F(x) is continuous at x = 5.

11. Suppose L_1 and L_2 are two different limits. Without loss of generality assume $L_2>L_1$. Let $\epsilon=\frac{1}{3}(L_2-L_1)$.

Since $\lim\limits_{x\to x_0} f(x)=L_1$ there is a $\delta_1>0$ such that $0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L_1|<\epsilon \Rightarrow -\epsilon<f(x)-L_1<\epsilon$

$\Rightarrow -\frac{1}{3}(L_2-L_1)+L_1<f(x)<\frac{1}{3}(L_2-L_1)+L_1 \Rightarrow 4L_1-L_2<3f(x)<2L_1+L_2$. Likewise, $\lim\limits_{x\to x_0} f(x)=L_2$

so there is a δ_2 such that $0<|x-x_0|<\delta_2 \Rightarrow |f(x)-L_2|<\epsilon \Rightarrow -\epsilon<f(x)-L_2<\epsilon$

$\Rightarrow -\frac{1}{3}(L_2-L_1)+L_2<f(x)<\frac{1}{3}(L_2-L_1)+L_2 \Rightarrow 2L_2+L_1<3f(x)<4L_2-L_1$

$\Rightarrow L_1-4L_2<-3f(x)<-2L_2-L_1$. If $\delta=\min\{\delta_1,\delta_2\}$ both inequalities must hold for $0<|x-x_0|<\delta$:

$$\left.\begin{array}{r}4L_1-L_2<\ \ 3f(x)<\ \ 2L_1+L_2\\ L_1-4L_2<-3f(x)<-2L_2-L_1\end{array}\right\} \Rightarrow 5(L_1-L_2)<0<L_1-L_2.$$ That is, $L_1-L_2<0$ <u>and</u> $L_1-L_2>0$,

a contradiction.

12. Suppose $\lim\limits_{x\to c} f(x)=L$. If $k=0$, then $\lim\limits_{x\to c} kf(x)=\lim\limits_{x\to c} 0=0=0\cdot\lim\limits_{x\to c} f(x)$ and we are done.

If $k\neq 0$, then given any $\epsilon>0$, there is a $\delta>0$ so that $0<|x-c|<\delta \Rightarrow |f(x)-L|<\frac{\epsilon}{|k|} \Rightarrow |k||f(x)-L|<\epsilon$

$\Rightarrow |k(f(x)-L)|<\epsilon \Rightarrow |(kf(x))-(kL)|<\epsilon$. Thus, $\lim\limits_{x\to c} kf(x)=kL=k\left(\lim\limits_{x\to c} f(x)\right)$.

13. (a) Since $x \to 0^+, 0 < x^3 < x < 1 \Rightarrow (x^3 - x) \to 0^- \Rightarrow \lim\limits_{x \to 0^+} f(x^3 - x) = \lim\limits_{y \to 0^-} f(y) = B$ where $y = x^3 - x$.

(b) Since $x \to 0^-, -1 < x < x^3 < 0 \Rightarrow (x^3 - x) \to 0^+ \Rightarrow \lim\limits_{x \to 0^-} f(x^3 - x) = \lim\limits_{y \to 0^+} f(y) = A$ where $y = x^3 - x$.

(c) Since $x \to 0^+, 0 < x^4 < x^2 < 1 \Rightarrow (x^2 - x^4) \to 0^+ \Rightarrow \lim\limits_{x \to 0^+} f(x^2 - x^4) = \lim\limits_{y \to 0^+} f(y) = A$ where $y = x^2 - x^4$.

(d) Since $x \to 0^-, -1 < x < 0 \Rightarrow 0 < x^4 < x^2 < 1 \Rightarrow (x^2 - x^4) \to 0^+ \Rightarrow \lim\limits_{x \to 0^+} f(x^2 - x^4) = A$ as in part (c).

14. (a) True, because if $\lim\limits_{x \to a} (f(x) + g(x))$ exists then $\lim\limits_{x \to a} (f(x) + g(x)) - \lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} [(f(x) + g(x)) - f(x)]$ $= \lim\limits_{x \to a} g(x)$ exists, contrary to assumption.

(b) False; for example take $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$. Then neither $\lim\limits_{x \to 0} f(x)$ nor $\lim\limits_{x \to 0} g(x)$ exists, but $\lim\limits_{x \to 0} (f(x) + g(x)) = \lim\limits_{x \to 0} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim\limits_{x \to 0} 0 = 0$ exists.

(c) True, because $g(x) = |x|$ is continuous $\Rightarrow g(f(x)) = |f(x)|$ is continuous (it is the composite of continuous functions).

(d) False; for example let $f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases} \Rightarrow f(x)$ is discontinuous at $x = 0$. However $|f(x)| = 1$ is continuous at $x = 0$.

15. Show $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim\limits_{x \to -1} \frac{(x + 1)(x - 1)}{(x + 1)} = -2, x \ne -1$.

Define the continuous extension of f(x) as $F(x) = \begin{cases} \frac{x^2 - 1}{x + 1}, & x \ne -1 \\ -2 & , x = -1 \end{cases}$. We now prove the limit of f(x) as $x \to -1$ exists and has the correct value.

Step 1: $\left|\frac{x^2 - 1}{x + 1} - (-2)\right| < \epsilon \Rightarrow -\epsilon < \frac{(x + 1)(x - 1)}{(x + 1)} + 2 < \epsilon \Rightarrow -\epsilon < (x - 1) + 2 < \epsilon, x \ne -1 \Rightarrow -\epsilon - 1 < x < \epsilon - 1$.

Step 2: $|x - (-1)| < \delta \Rightarrow -\delta < x + 1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$.

Then $-\delta - 1 = -\epsilon - 1 \Rightarrow \delta = \epsilon$, or $\delta - 1 = \epsilon - 1 \Rightarrow \delta = \epsilon$. Choose $\delta = \epsilon$. Then $0 < |x - (-1)| < \delta$ $\Rightarrow \left|\frac{x^2 - 1}{x + 1} - (-2)\right| < \epsilon \Rightarrow \lim\limits_{x \to -1} F(x) = -2$. Since the conditions of the continuity test are met by F(x), then f(x) has a continuous extension to F(x) at $x = -1$.

16. Show $\lim\limits_{x \to 3} g(x) = \lim\limits_{x \to 3} \frac{x^2 - 2x - 3}{2x - 6} = \lim\limits_{x \to 3} \frac{(x - 3)(x + 1)}{2(x - 3)} = 2, x \ne 3$.

Define the continuous extension of g(x) as $G(x) = \begin{cases} \frac{x^2 - 2x - 3}{2x - 6}, & x \ne 3 \\ 2 & , x = 3 \end{cases}$. We now prove the limit of g(x) as $x \to 3$ exists and has the correct value.

Step 1: $\left|\frac{x^2 - 2x - 3}{2x - 6} - 2\right| < \epsilon \Rightarrow -\epsilon < \frac{(x - 3)(x + 1)}{2(x - 3)} - 2 < \epsilon \Rightarrow -\epsilon < \frac{x + 1}{2} - 2 < \epsilon, x \ne 3 \Rightarrow 3 - 2\epsilon < x < 3 + 2\epsilon$.

Step 2: $|x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow 3 - \delta < x < \delta + 3$.

Then, $3 - \delta = 3 - 2\epsilon \Rightarrow \delta = 2\epsilon$, or $\delta + 3 = 3 + 2\epsilon \Rightarrow \delta = 2\epsilon$. Choose $\delta = 2\epsilon$. Then $0 < |x - 3| < \delta$ $\Rightarrow \left|\frac{x^2 - 2x - 3}{2x - 6} - 2\right| < \epsilon \Rightarrow \lim\limits_{x \to 3} \frac{(x - 3)(x + 1)}{2(x - 3)} = 2$. Since the conditions of the continuity test hold for G(x), g(x) can be continuously extended to G(x) at $x = 3$.

17. (a) Let $\epsilon > 0$ be given. If x is rational, then $f(x) = x \Rightarrow |f(x) - 0| = |x - 0| < \epsilon \Leftrightarrow |x - 0| < \epsilon$; i.e., choose $\delta = \epsilon$. Then $|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$ for x rational. If x is irrational, then $f(x) = 0 \Rightarrow |f(x) - 0| < \epsilon$ $\Leftrightarrow 0 < \epsilon$ which is true no matter how close irrational x is to 0, so again we can choose $\delta = \epsilon$. In either case, given $\epsilon > 0$ there is a $\delta = \epsilon > 0$ such that $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$. Therefore, f is continuous at $x = 0$.

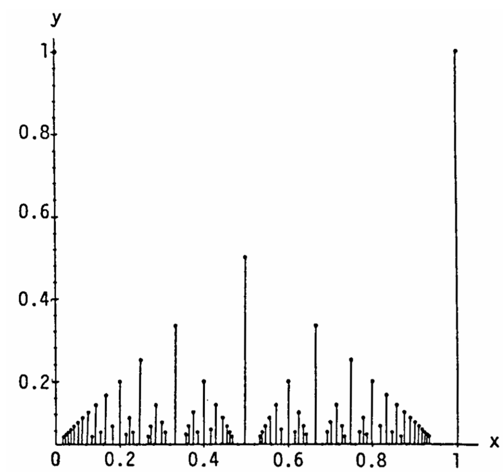
(b) Choose $x = c > 0$. Then within any interval $(c - \delta, c + \delta)$ there are both rational and irrational numbers. If c is rational, pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is an irrational number x in $(c - \delta, c + \delta) \Rightarrow |f(x) - f(c)| = |0 - c| = c > \frac{c}{2} = \epsilon$. That is, f is not continuous at any rational $c > 0$. On the other hand, suppose c is irrational $\Rightarrow$ $f(c) = 0$. Again pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is a rational number x in $(c - \delta, c + \delta)$ with $|x - c| < \frac{c}{2} = \epsilon \Leftrightarrow \frac{c}{2} < x < \frac{3c}{2}$. Then $|f(x) - f(c)| = |x - 0|$ $= |x| > \frac{c}{2} = \epsilon \Rightarrow$ f is not continuous at any irrational $c > 0$.

If $x = c < 0$, repeat the argument picking $\epsilon = \frac{|c|}{2} = \frac{-c}{2}$. Therefore f fails to be continuous at any nonzero value $x = c$.

18. (a) Let $c = \frac{m}{n}$ be a rational number in $[0, 1]$ reduced to lowest terms $\Rightarrow$ $f(c) = \frac{1}{n}$. Pick $\epsilon = \frac{1}{2n}$. No matter how small $\delta > 0$ is taken, there is an irrational number x in the interval $(c - \delta, c + \delta) \Rightarrow |f(x) - f(c)| = \left|0 - \frac{1}{n}\right|$ $= \frac{1}{n} > \frac{1}{2n} = \epsilon$. Therefore f is discontinuous at $x = c$, a rational number.

(b) Now suppose c is an irrational number $\Rightarrow$ $f(c) = 0$. Let $\epsilon > 0$ be given. Notice that $\frac{1}{2}$ is the only rational number reduced to lowest terms with denominator 2 and belonging to $[0, 1]$; $\frac{1}{3}$ and $\frac{2}{3}$ the only rationals with denominator 3 belonging to $[0, 1]$; $\frac{1}{4}$ and $\frac{3}{4}$ with denominator 4 in $[0, 1]$; $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ and $\frac{4}{5}$ with denominator 5 in $[0, 1]$; etc. In general, choose N so that $\frac{1}{N} < \epsilon \Rightarrow$ there exist only finitely many rationals in $[0, 1]$ having denominator $\le N$, say $r_1, r_2, \ldots, r_p$. Let $\delta = \min\{|c - r_i| : i = 1, \ldots, p\}$. Then the interval $(c - \delta, c + \delta)$ contains no rational numbers with denominator $\le N$. Thus, $0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| = |f(x) - 0|$ $= |f(x)| \le \frac{1}{N} < \epsilon \Rightarrow$ f is continuous at $x = c$ irrational.

(c) The graph looks like the markings on a typical ruler when the points $(x, f(x))$ on the graph of f(x) are connected to the x-axis with vertical lines.



$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

19. Yes. Let R be the radius of the equator (earth) and suppose at a fixed instant of time we label noon as the zero point, 0, on the equator $\Rightarrow$ $0 + \pi R$ represents the midnight point (at the same exact time). Suppose x_1 is a point on the equator "just after" noon $\Rightarrow$ $x_1 + \pi R$ is simultaneously "just after" midnight. It seems reasonable that the temperature T at a point just after noon is hotter than it would be at the diametrically opposite point just after midnight: That is, $T(x_1) - T(x_1 + \pi R) > 0$. At exactly the same moment in time pick x_2 to be a point just before midnight $\Rightarrow$ $x_2 + \pi R$ is just before noon. Then $T(x_2) - T(x_2 + \pi R) < 0$. Assuming the temperature function T is continuous along the equator (which is reasonable), the Intermediate Value Theorem says there is a point c between 0 (noon) and πR (simultaneously midnight) such that $T(c) - T(c + \pi R) = 0$; i.e., there is always a pair of antipodal points on the earth's equator where the temperatures are the same.

20. $\lim\limits_{x \to c} f(x)g(x) = \lim\limits_{x \to c} \frac{1}{4}\left[(f(x) + g(x))^2 - (f(x) - g(x))^2\right] = \frac{1}{4}\left[\left(\lim\limits_{x \to c}(f(x) + g(x))\right)^2 - \left(\lim\limits_{x \to c}(f(x) - g(x))\right)^2\right]$
$= \frac{1}{4}\left(3^2 - (-1)^2\right) = 2.$

21. (a) At $x = 0$: $\lim\limits_{a \to 0} r_+(a) = \lim\limits_{a \to 0} \frac{-1+\sqrt{1+a}}{a} = \lim\limits_{a \to 0} \left(\frac{-1+\sqrt{1+a}}{a}\right)\left(\frac{-1-\sqrt{1+a}}{-1-\sqrt{1+a}}\right)$

$= \lim\limits_{a \to 0} \frac{1-(1+a)}{a\left(-1-\sqrt{1+a}\right)} = \frac{-1}{-1-\sqrt{1+0}} = \frac{1}{2}$

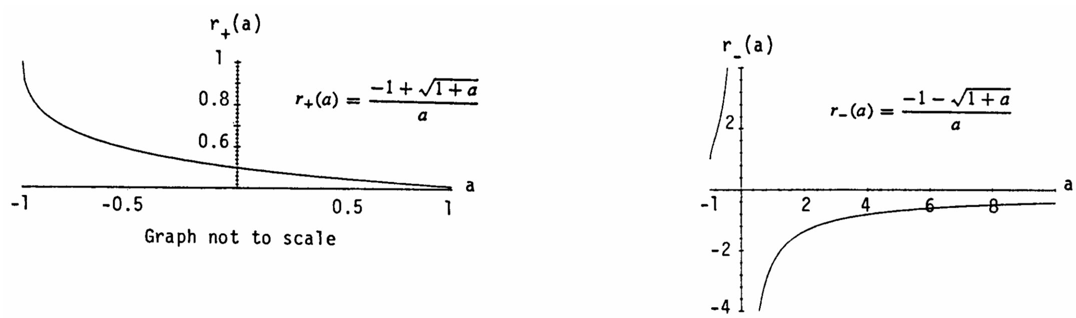
At $x = -1$: $\lim\limits_{a \to -1^+} r_+(a) = \lim\limits_{a \to -1^+} \frac{1-(1+a)}{a\left(-1-\sqrt{1+a}\right)} = \lim\limits_{a \to -1} \frac{-a}{a\left(-1-\sqrt{1+a}\right)} = \frac{-1}{-1-\sqrt{0}} = 1$

(b) At $x = 0$: $\lim\limits_{a \to 0^-} r_-(a) = \lim\limits_{a \to 0^-} \frac{-1-\sqrt{1+a}}{a} = \lim\limits_{a \to 0^-} \left(\frac{-1-\sqrt{1+a}}{a}\right)\left(\frac{-1+\sqrt{1+a}}{-1+\sqrt{1+a}}\right)$

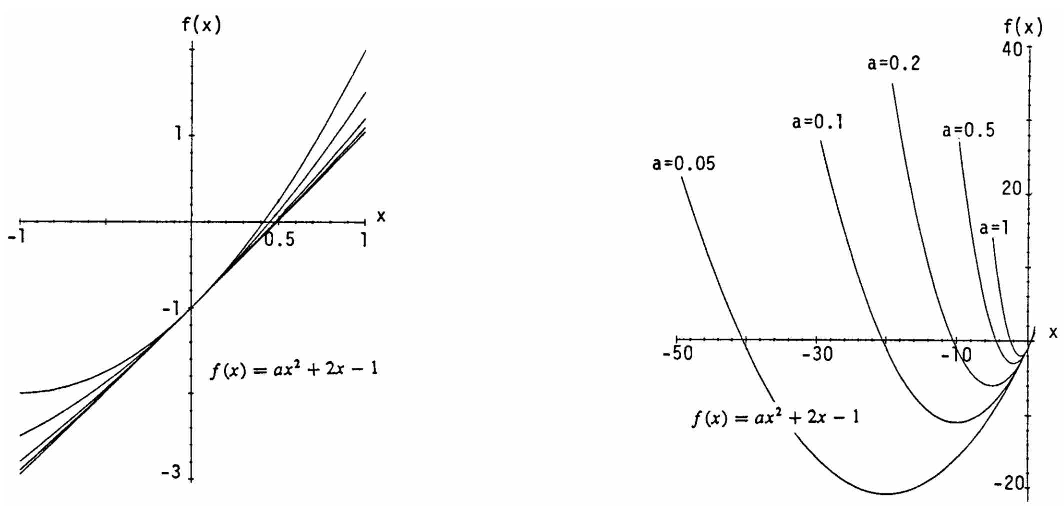
$= \lim\limits_{a \to 0^-} \frac{1-(1+a)}{a\left(-1+\sqrt{1+a}\right)} = \lim\limits_{a \to 0^-} \frac{-a}{a\left(-1+\sqrt{1+a}\right)} = \lim\limits_{a \to 0^-} \frac{-1}{-1+\sqrt{1+a}} = \infty$ (because the denominator is always negative); $\lim\limits_{a \to 0^+} r_-(a) = \lim\limits_{a \to 0^+} \frac{-1}{-1+\sqrt{1+a}} = -\infty$ (because the denominator is always positive). Therefore, $\lim\limits_{a \to 0} r_-(a)$ does not exist.

At $x = -1$: $\lim\limits_{a \to -1^+} r_-(a) = \lim\limits_{a \to -1^+} \frac{-1-\sqrt{1+a}}{a} = \lim\limits_{a \to -1^+} \frac{-1}{-1+\sqrt{1+a}} = 1$

(c)



(d)



22. $f(x) = x + 2\cos x \Rightarrow f(0) = 0 + 2\cos 0 = 2 > 0$ and $f(-\pi) = -\pi + 2\cos(-\pi) = -\pi - 2 < 0$. Since f(x) is continuous on $[-\pi, 0]$, by the Intermediate Value Theorem, f(x) must take on every value between $[-\pi - 2, 2]$. Thus there is some number c in $[-\pi, 0]$ such that $f(c) = 0$; i.e., c is a solution to $x + 2\cos x = 0$.

23. (a) The function f is bounded on D if $f(x) \ge M$ and $f(x) \le N$ for all x in D. This means $M \le f(x) \le N$ for all x in D. Choose B to be $\max\{|M|, |N|\}$. Then $|f(x)| \le B$. On the other hand, if $|f(x)| \le B$, then $-B \le f(x) \le B \Rightarrow f(x) \ge -B$ and $f(x) \le B \Rightarrow f(x)$ is bounded on D with $N = B$ an upper bound and $M = -B$ a lower bound.

(b) Assume $f(x) \le N$ for all x and that $L > N$. Let $\epsilon = \frac{L-N}{2}$. Since $\lim\limits_{x \to x_0} f(x) = L$ there is a $\delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon \Leftrightarrow L - \epsilon < f(x) < L + \epsilon \Leftrightarrow L - \frac{L-N}{2} < f(x) < L + \frac{L-N}{2}$ $\Leftrightarrow \frac{L+N}{2} < f(x) < \frac{3L-N}{2}$. But $L > N \Rightarrow \frac{L+N}{2} > N \Rightarrow N < f(x)$ contrary to the boundedness assumption $f(x) \le N$. This contradiction proves $L \le N$.

(c) Assume $M \le f(x)$ for all x and that $L < M$. Let $\epsilon = \frac{M-L}{2}$. As in part (b), $0 < |x - x_0| < \delta$ $\Rightarrow L - \frac{M-L}{2} < f(x) < L + \frac{M-L}{2} \Leftrightarrow \frac{3L-M}{2} < f(x) < \frac{M+L}{2} < M$, a contradiction.

24. (a) If $a \ge b$, then $a - b \ge 0 \Rightarrow |a - b| = a - b \Rightarrow \max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2} = \frac{a+b}{2} + \frac{a-b}{2} = \frac{2a}{2} = a$.
If $a \le b$, then $a - b \le 0 \Rightarrow |a - b| = -(a - b) = b - a \Rightarrow \max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2} = \frac{a+b}{2} + \frac{b-a}{2}$ $= \frac{2b}{2} = b$.
(b) Let $\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$.

25. $\lim\limits_{x \to 0} = \frac{\sin(1-\cos x)}{x} = \lim\limits_{x \to 0} \frac{\sin(1-\cos x)}{1-\cos x} \cdot \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \lim\limits_{x \to 0} \frac{\sin(1-\cos x)}{1-\cos x} \cdot \lim\limits_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = 1 \cdot \lim\limits_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$
$= \lim\limits_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} = 1 \cdot \left(\frac{0}{2}\right) = 0.$

26. $\lim\limits_{x \to 0^+} \frac{\sin x}{\sin \sqrt{x}} = \lim\limits_{x \to 0^+} \frac{\sin x}{x} \cdot \frac{\sqrt{x}}{\sin \sqrt{x}} \cdot \frac{x}{\sqrt{x}} = 1 \cdot \lim\limits_{x \to 0^+} \frac{1}{\left(\frac{\sin \sqrt{x}}{\sqrt{x}}\right)} \cdot \lim\limits_{x \to 0^+} \sqrt{x} = 1 \cdot 0 \cdot 0 = 0.$

27. $\lim\limits_{x \to 0} \frac{\sin(\sin x)}{x} = \lim\limits_{x \to 0} \frac{\sin(\sin x)}{\sin x} \cdot \frac{\sin x}{x} = \lim\limits_{x \to 0} \frac{\sin(\sin x)}{\sin x} \cdot \lim\limits_{x \to 0} \frac{\sin x}{x} = 1 \cdot 1 = 1.$

28. $\lim\limits_{x \to 0} \frac{\sin(x^2+x)}{x} = \lim\limits_{x \to 0} \frac{\sin(x^2+x)}{x^2+x} \cdot (x+1) = \lim\limits_{x \to 0} \frac{\sin(x^2+x)}{x^2+x} \cdot \lim\limits_{x \to 0} (x+1) = 1 \cdot 1 = 1$

29. $\lim\limits_{x \to 2} \frac{\sin(x^2-4)}{x-2} = \lim\limits_{x \to 2} \frac{\sin(x^2-4)}{x^2-4} \cdot (x+2) = \lim\limits_{x \to 2} \frac{\sin(x^2-4)}{x^2-4} \cdot \lim\limits_{x \to 2} (x+2) = 1 \cdot 4 = 4$

30. $\lim\limits_{x \to 9} \frac{\sin(\sqrt{x}-3)}{x-9} = \lim\limits_{x \to 9} \frac{\sin(\sqrt{x}-3)}{\sqrt{x}-3} \cdot \frac{1}{\sqrt{x}+3} = \lim\limits_{x \to 9} \frac{\sin(\sqrt{x}-3)}{\sqrt{x}-3} \cdot \lim\limits_{x \to 9} \frac{1}{\sqrt{x}+3} = 1 \cdot \frac{1}{6} = \frac{1}{6}$

31. Since the highest power of x in the numerator is 1 more than the highest power of x in the denominator, there is an oblique asymptote. $y = \frac{2x^{3/2}+2x-3}{\sqrt{x}+1} = 2x - \frac{3}{\sqrt{x}+1}$, thus the oblique asymptote is $y = 2x$.

32. As $x \to \pm\infty$, $\frac{1}{x} \to 0 \Rightarrow \sin\left(\frac{1}{x}\right) \to 0 \Rightarrow 1 + \sin\left(\frac{1}{x}\right) \to 1$, thus as $x \to \pm\infty$, $y = x + x\sin\left(\frac{1}{x}\right) = x\left(1 + \sin\left(\frac{1}{x}\right)\right) \to x$; thus the oblique asymptote is $y = x$.

33. As $x \to \pm\infty$, $x^2 + 1 \to x^2 \Rightarrow \sqrt{x^2+1} \to \sqrt{x^2}$; as $x \to -\infty$, $\sqrt{x^2} = -x$, and as $x \to +\infty$, $\sqrt{x^2} = x$; thus the oblique asymptotes are $y = x$ and $y = -x$.

34. As $x \to \pm\infty$, $x + 2 \to x \Rightarrow \sqrt{x^2+2x} = \sqrt{x(x+2)} \to \sqrt{x^2}$; as $x \to -\infty$, $\sqrt{x^2} = -x$, and as $x \to +\infty$, $\sqrt{x^2} = x$; asymptotes are $y = x$ and $y = -x$.